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CONVERGENT NETS OF PARABOLIC AND GENERALIZED SUPERPARABOLIC FUNCTIONS

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ABSTRACT. The well-known convergence properties of families of harmonic functions are generalized to functions which satisfy $Lu = 0$ where L is the weak parabolic operator in divergence form. Properties of superharmonic functions are obtained for generalized superparabolic functions. These results are obtained on any bounded domain in E^{n+1}.

Consider the parabolic operator in divergence form given by

$$Lu = u_t - \{a_{ij}(x, t)u,_i + d_j(x, t)u\},_j - b_j(x, t)u,_j - c(x, t)u.$$

In two preceding papers by the author [4], [5], existence, representation, and a maximum principle were obtained for solutions of $Lu = 0$ in the cylindrical domain $Q = \Omega \times (0, T)$ for $\Omega \subset E^n$, and generalized superparabolic functions in Q were introduced. In this article the author will consider convergent nets of parabolic and superparabolic functions on a bounded domain U assumed to be in $E^n \times (0, T)$. Since some of the properties obtained in this article will depend on results for superparabolic functions, it is necessary to restate these results for the domain U. The numbering of definitions and theorems in this article continues from those in [5].

Definition 7. Let $z = (x, t)$, $w = (y, s) \in U$. $z \prec w$ in U if there is a polygonal path $\{C_{z,w}(\alpha), 0 \leq \alpha \leq 1\}$ such that

(1) $C_{z,w}(0) = \{z\}$, $C_{z,w}(1) = \{w\}$,

(2) if $C_{z,w}(\alpha) = \{(\xi_\alpha, \tau_\alpha)\}$, then $\alpha < \beta$ implies $\tau_\alpha \leq \tau_\beta$,

(3) $\{C_{z,w}(\alpha); 0 \leq \alpha \leq 1\} \subset U$.

Note that Definitions 4, 5, and 6 can be generalized to the domain U since all properties are in terms of standard rectangles in the given domain. Accordingly, let $\mathfrak{L}_U$, $\mathfrak{S}'_U$, $\mathfrak{S}''_U$, and $\mathfrak{S}'''_U$ denote the corresponding spaces. The

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theorems and corollaries stated below correspond to the analogous results obtained for Q in [5].

Theorem 4′. *If $u \in \mathcal{S}_U'''$ and if there is a $z_0 \in U$ such that $0 \geq u(z_0) = \inf_U u$, then $u(z) = u(z_0)$ for all $z \prec z_0$ in U.*

The following corollary is the analogue of that preceding Theorem 7 in [5].

Corollary. *If $u \in \mathcal{S}_U'$, then*

(1) *$u(z_0) < \infty$ implies $u(z) < \infty$ for all $z \prec z_0$ in U;*

(2) *$u(z_0) = \infty$ implies $u(z) = \infty$ for all $z_0 \prec z$ in U.*

Theorem 10′. *Let $u \in \mathcal{S}_U'$ and let R be a standard rectangle in U. If $u < \infty$ on R, then $L(u; z, R)$ exists and the function*

$$v(z) = \begin{cases} u(z), & z \in U - R, \\ L(u; z, R), & z \in R, \end{cases}$$

satisfies $u \geq v$, $Lv = 0$ on R, and $v \in \mathcal{S}_U'$.

Lemma 2. *If $\{u_\alpha; \alpha \in A\}$ is a net of functions parabolic on U which converges uniformly on U to u, then u is parabolic on U.*

Proof. For any standard rectangle R, $\overline{R} \subset U$, and any $z \in R$, $u_\alpha(z) = L(u_\alpha; z, R)$. Since $u_\alpha \to u$ uniformly on R, it follows that $u(z) = L(u; z, R)$. Since R in U was arbitrary, u is parabolic on U.

Theorem 11. *If $\mathcal{F}$ is a family of parabolic functions on U uniformly bounded on a neighborhood of each point of U and if K is compact in U, then the family $\mathcal{F}$ is uniformly equicontinuous on K and each net $\{u_\alpha; \alpha \in A\}$ in $\mathcal{F}$ has a subnet which converges uniformly on K. If $\{u_\alpha; \alpha \in A\}$ is a convergent net of uniformly bounded parabolic functions on U, then $u \equiv \lim_A u_\alpha$ is parabolic on U.*

Proof. For each $z \in K$, let N_z denote the neighborhood of z on which the family $\mathcal{F}$ is uniformly bounded. Then $\bigcup\{N_z; z \in K\}$ is an open cover for K and, by the Heine-Borel theorem, there is a finite subcover N. Let $N = \bigcup_{j=1}^{k}\{N_z; z = z_j \in K\}$. Then the family $\mathcal{F}$ is uniformly bounded on N (and hence on K). Each member of $\mathcal{F}$ is parabolic on U and, hence, is Hölder continuous on U. Since K is compact in U, the Hölder coefficients and exponents can be made uniform for the family $\mathcal{F}$ and it follows that $\mathcal{F}$ is an equicontinuous family on K. It follows from Arzela's theorem that each net $\{u_\alpha; \alpha \in A\}$ in $\mathcal{F}$ has a uniformly convergent subnet on K.

Let $\{u_\alpha; \alpha \in K\}$ be a convergent net of uniformly bounded parabolic

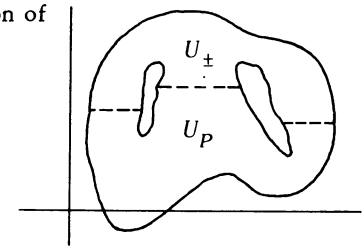
functions on U. Then $u \equiv \lim_A u_\alpha$ exists. Let $\{K_j\}$ be a sequence of compact sets in U such that $K_j \uparrow U$. It follows from what has been shown that for K_1 there is a subnet $\{u_\alpha\}_1$ of $\{u_\alpha;\ \alpha \in A\}$ which converges to a parabolic function on K_1. Since $u_\alpha \to u$ on U, u is parabolic on K_1. Using what has been shown on K_2, there is a subnet $\{u_\alpha\}_2$ of $\{u_\alpha\}_1$ which converges to a parabolic function, u, on K_2. Continuing this process, the diagonalization process can be used to see that u is parabolic on all compact subsets of U and, hence, on U itself.

Definition 8. Let u be defined on U. Let $U_P = \{z \in U;\ u$ locally parabolic at $z\}$ and $U_\pm = \{z \in U;\ u(z) = \pm\infty\}$. u is said to have property $P_\pm$ if

(1) $w \in U_P$ implies $z \in U_P$ for all $z \in U$ with $z \prec w$.

(2) $w \in U_\pm$ implies $z \in U_\pm$ for all $z \in U$ with $z \succ w$.

(3) $U - U_P - U_+ - U_-$ is a finite union of sets of the form $\Omega_j \times (t_j)$.



Theorem 12. *If u_j is an increasing (or decreasing) sequence of parabolic functions on U, then $u(z) = \lim u_j(z)$ satisfies property P_+ (or P_-).*

Proof. Assume $u_j \uparrow u$. If for some $w \in U$, $u(w) < \infty$, then for any standard rectangle R with $\bar{R} \subset U$ and $w \in R$, the monotone convergence theorem implies that

$$L(u;\ w,\ R) = \lim L(u_j;\ w,\ R) = \lim u_j(w) = u(w) < \infty \tag{1}$$

and, hence, $u(z) < \infty$ on $\partial_p R$ with $z \prec w$. Therefore, U_P satisfies property (1) of Definition 8. Since $u \geq u_1 \geq -\infty$ by the parabolicity of u_1, $U_- = \emptyset$. To see that U_+ satisfies property (2), suppose $u(w) = +\infty$ and suppose for some $z \in U$ with $w \prec z$ that $u(z) < \infty$. Then by our first argument $u(w) < \infty$ and a contradiction is obtained. Therefore property (2) is satisfied. Since $u_j \uparrow u$, for any $z \in U$ either $u(z) < \infty$ or $u(z) = \infty$ must hold. Hence, property (3) is vacously satisfied.

Definition 9. A family $\mathcal{F}$ of functions defined on U is *right-directed* if for each pair $u, v \in \mathcal{F}$ there is a function $w \in \mathcal{F}$ such that $u \leq w$ and $v \leq w$. $\mathcal{F}$ is *left-directed* if the above inequalities are reversed.

Lemma 3. *If $\{u_i;\ i \in I\}$ is a right-directed (left-directed) family of functions parabolic on U, then $u \equiv \sup_I u_i$ ($u \equiv \inf_I u_i$) satisfies property P_+ (P_-) in U.*

Proof. If $\{u_i;\ i \in I\}$ contains only one function then we are done. If

$\{u_i;\ i \in I\}$ contains two or more functions, then the right-directedness can be used to obtain a sequence $\{u_j\}$ in $\{u_i;\ i \in I\}$ such that $u_j \uparrow u \equiv \sup_I u_i$. The desired result follows from Theorem 12.

Definition 10. A family $\mathcal{F}$ in $\mathcal{S}'_U$ is *saturated* if

(1) $u,\ v \in \mathcal{F} \Rightarrow u \wedge v \in \mathcal{F}$,

(2) $u \in \mathcal{F} \Rightarrow u^* \in \mathcal{F}$ where, for some standard rectangle R in U,

$$u^*(z) = \begin{cases} u(z) & \text{in } U - R, \\ L(u;\ z,\ R) & \text{in } R. \end{cases} \tag{2}$$

Theorem 13. *If $\mathcal{F}$ is a saturated family in $\mathcal{S}'_U$, then $v \equiv \inf_{\mathcal{F}} u$ satisfies property P_- in U.*

Proof. Let R be an arbitrary standard rectangle. It will be shown that v satisfies property P_- on R and the desired result will follow from the arbitrariness of R in U.

For each $u \in \mathcal{F}$, let u^* be defined as in (2). By Theorem 10′, $u^* \leq u$ on R. Moreover, since $\mathcal{F}$ is saturated in $\mathcal{S}'_U$, $u \in \mathcal{F}$ implies $u^* \in \mathcal{F}$. Therefore,

$$v = \inf\{u;\ u \in \mathcal{F}\} = \inf\{u^*;\ u \in \mathcal{F}\}.$$

Let $\mathcal{F}^* = \{u^*;\ u \in \mathcal{F}\}$. If it can be shown that $\mathcal{F}^*$ is a left-directed family, the desired result will follow from Lemma 3. Let $u,\ w \in \mathcal{F}$. Then $u \wedge w \in \mathcal{F}$ and hence u^*, w^*, and $(u \wedge w)^* \in \mathcal{F}$. However $u \wedge w \leq u$ implies $(u \wedge w)^* \leq u^*$ and similarly $(u \wedge w)^* \leq w^*$. That is, for any two elements $u^*,\ w^* \in \mathcal{F}^*$, there is an element, namely $(u \wedge w)^*$, in $\mathcal{F}$ such that $(u \wedge w)^* \leq u^*$ and $(u \wedge w)^* \leq w^*$. Hence, $\mathcal{F}^*$ is left-directed and the proof is complete.

The preceding results for families of parabolic and generalized superparabolic functions will next be used to find the greatest parabolic minorant of a given function u on U. Let W be an open set in U and let $u \in \mathcal{S}'_U$. Let $\{R_j\}$ be a sequence of standard rectangles satisfying

(1) $\overline{R}_j \subset W$ for all j,

(2) $W = \bigcup_{j=1} R_j$,

(3) each rectangle R_j occurs infinitely often in the sequence $\{R_j\}$.

Define

$$u_1(z) = \begin{cases} u(z), & z \in U - R_1, \\ L(u;\ z,\ R_1), & z \in R_1, \end{cases}$$

and inductively

$$u_n(z) = \begin{cases} u_{n-1}(z), & z \in U - R_n, \\ L(u_{n-1}; z, R_n), & z \in R_n. \end{cases}$$

Then by Theorem 10′ $u_n \leq u_{n-1}$ on U, u_n is parabolic on R_n, and $u_n \in \mathcal{S}_U'$. Since $\{u_n\}$ is a decreasing sequence, $u_\infty(z) \equiv \lim_{n\to\infty} u_n(z)$ satisfies property P_- on U.

Definition 11. u_∞ is called the *reduction of u over W relative to U.*

It appears as if for a given open set W in U the corresponding reduction u_∞ will depend on the defining sequence $\{R_j\}$. It will follow from Theorem 15 that u_∞ is independent of the choice of $\{R_j\}$ if there is a parabolic function v on W with $v \leq u$.

Theorem 14. *If $u \in \mathcal{S}_U'$, W is open in U, and u_∞ is the reduction of u over W relative to U corresponding to the sequence $\{R_j\}$, then u_∞ satisfies property P_- on each component of W, and $u_\infty = u$ on $U - W$.*

Proof. Without loss of generality assume W is connected. For each j, there is a sequence $\{j_k\}$ such that $R_{j_k} = R_j$ for all k. Thus, on R_j,

$$u_\infty(z) = \lim_{k\to\infty} u_{j_k}(z).$$

Since u_{j_k} is parabolic on R_j for all k, it follows from Theorem 12 that u_∞ satisfies property P_- on R_j. Using this procedure on each different rectangle R_j in the defining sequence, it follows that u_∞ satisfies property P_- on W. Since $u_j \equiv u$ on $U - W$ for all j, $u_\infty = u$ on $U - W$.

Lemma 4. *If $u \in \mathcal{S}_U'$, W is open in U, v is parabolic on W, and $v \leq u$ on W, then $v \leq u_\infty \leq u$ on W where u_∞ is the reduction of u over W in U.*

Proof. Let $\{R_j\}$ be the sequence of standard rectangles which define the reduction of u over W, u_∞. Then

$$v(z) = L(v; z, R_1) \leq L(u; z, R_1) = u_1(z) \leq u(z) \quad \text{on } R_1,$$

and

$$v(z) \leq u(z) = u_1(z) \quad \text{on } W - R_1.$$

Therefore, $v \leq u_1 \leq u$ on W. Proceeding inductively, the desired result is obtained.

Definition 12. If $u \in \mathcal{S}_U'$, v is parabolic on U, and $v \leq u$ on U, then v is called a *parabolic minorant* of u. v is the *greatest parabolic minorant* of u if v is a parabolic minorant of u and any other parabolic minorant of u is $\leq v$.

Theorem 15. *If $v \in \mathcal{S}'_U$, W is open in U, and u has a parabolic minorant v on W, then u has a unique greatest parabolic minorant on W; namely, u_∞.*

Proof. It follows from Lemma 4 that $v \leq u_\infty \leq u$ on W. But v parabolic on W implies $v > -\infty$ on W. Therefore, since u_∞ satisfies property P_- on W, u_∞ is parabolic on W. If there were two such u_∞, say $u_{\infty,1}$ and $u_{\infty,2}$, then $u_{\infty,1} \leq u_{\infty,2} \leq u$ on W since $u_{\infty,1}$ is a parabolic minorant and $u_{\infty,2} \leq u_{\infty,1} \leq u$ on W since $u_{\infty,2}$ is a parabolic minorant. Therefore, u_∞ is unique and it is the greatest parabolic minorant of u.

Theorem 16. *If $u, v \in \mathcal{S}'_U$ have harmonic minorants, then $(u+v)_\infty = u_\infty + v_\infty$.*

Proof. Since u and v have harmonic minorants, $u+v$ does also and, hence, u_∞, v_∞, and $(u+v)_\infty$ are independent of the defining sequence $\{R_j\}$. For such a sequence and $j \geq 1$

$$(u+v)_j(x) = \begin{cases} (u+v)_{j-1}(z) & \text{on } U - R_j, \\ L((u+v)_{j-1}; z, R_j) & \text{on } R_j, \end{cases}$$

$$= \left\{\begin{matrix} u_{j-1}(z) & \text{on } U - R_j \\ L(u_{j-1}; z, R_j) & \text{on } R_j \end{matrix}\right\} + \left\{\begin{matrix} v_{j-1}(z) & \text{on } U - R_j \\ L(v_{j-1}; z, R_j) & \text{on } R_j \end{matrix}\right\}$$

$$= u_j(z) + v_j(z).$$

Since $u_j \downarrow u_\infty$ and $v_j \downarrow v_\infty$, it follows that $(u+v)_j \downarrow (u+v)_\infty$ and $(u+v)_\infty = u_\infty + v_\infty$.

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