CONVERGENT NETS OF PARABOLIC
AND GENERALIZED SUPERPARABOLIC FUNCTIONS

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ABSTRACT. The well-known convergence properties of families of harmonic functions are generalized to functions which satisfy $L u = 0$ where $L$ is the weak parabolic operator in divergence form. Properties of superharmonic functions are obtained for generalized superparabolic functions. These results are obtained on any bounded domain in $E^{n+1}$.

Consider the parabolic operator in divergence form given by

$$L u = u_t - \sum_{j} a_{ij}(x, t)u_{x_i} + \sum_{j} b_j(x, t)u_{x_j} - c(x, t)u.$$

In two preceding papers by the author [4], [5], existence, representation, and a maximum principle were obtained for solutions of $L u = 0$ in the cylindrical domain $Q = \Omega \times (0, T)$ for $\Omega \subset E^n$, and generalized superparabolic functions in $Q$ were introduced. In this article the author will consider convergent nets of parabolic and superparabolic functions on a bounded domain $U$ assumed to be in $E^n \times (0, T)$. Since some of the properties obtained in this article will depend on results for superparabolic functions, it is necessary to restate these results for the domain $U$. The numbering of definitions and theorems in this article continues from those in [5].

Definition 7. Let $z = (x, t)$, $w = (y, s) \in U$. $z < w$ in $U$ if there is a polygonal path $\{C_{z,w}(a), 0 \leq a \leq 1\}$ such that

1. $C_{z,w}(0) = \{z\}, C_{z,w}(1) = \{w\}$,
2. if $C_{z,w}(a) = \{(\xi_a, \tau_a)\}$, then $\alpha < \beta$ implies $\tau_a < \tau_\beta$,
3. $\{C_{z,w}(a); 0 \leq a \leq 1\} \subset U$.

Note that Definitions 4, 5, and 6 can be generalized to the domain $U$ since all properties are in terms of standard rectangles in the given domain. Accordingly, let $U$, $U'$, $U''$, and $U'''$ denote the corresponding spaces. The

Received by the editors April 3, 1974.


Key words and phrases. Parabolic operator, sequences, generalized superparabolic functions, parabolic minorants.
Theorems and corollaries stated below correspond to the analogous results obtained for $Q$ in [5].

Theorem 4'. If $u \in S_U^m$ and if there is a $z_0 \in U$ such that $0 \geq u(z_0) = \inf_U u$, then $u(z) = u(z_0)$ for all $z < z_0$ in $U$.

The following corollary is the analogue of that preceding Theorem 7 in [5].

Corollary. If $u \in S_U^t$, then

1. $u(z_0) < \infty$ implies $u(z) < \infty$ for all $z < z_0$ in $U$;
2. $u(z_0) = \infty$ implies $u(z) = \infty$ for all $z < z$ in $U$.

Theorem 10'. Let $u \in S_U^t$ and let $R$ be a standard rectangle in $U$. If $u < \infty$ on $R$, then $L(u; z, R)$ exists and the function

$$v(z) = \begin{cases} u(z), & z \in U - R, \\ L(u; z, R), & z \in R, \end{cases}$$

satisfies $u \geq v$, $Lv = 0$ on $R$, and $v \in S_U^t$.

Lemma 2. If $\{u_\alpha; \alpha \in A\}$ is a net of functions parabolic on $U$ which converges uniformly on $U$ to $u$, then $u$ is parabolic on $U$.

Proof. For any standard rectangle $R$, $\bar{R} \subset U$, and any $z \in R$, $u_\alpha(z) = L(u_\alpha; z, R)$. Since $u_\alpha \to u$ uniformly on $R$, it follows that $u(z) = L(u; z, R)$. Since $R$ in $U$ was arbitrary, $u$ is parabolic on $U$.

Theorem 11. If $\mathcal{F}$ is a family of parabolic functions on $U$ uniformly bounded on each point of $U$ and if $K$ is compact in $U$, then the family $\mathcal{F}$ is uniformly equicontinuous on $K$ and each net $\{u_\alpha; \alpha \in A\}$ in $\mathcal{F}$ has a subnet which converges uniformly on $K$. If $\{u_\alpha; \alpha \in A\}$ is a convergent net of uniformly bounded parabolic functions on $U$, then $u = \lim A u_\alpha$ is parabolic on $U$.

Proof. For each $z \in K$, let $N_z$ denote the neighborhood of $z$ on which the family $\mathcal{F}$ is uniformly bounded. Then $\bigcup \{N_z; z \in K\}$ is an open cover for $K$ and, by the Heine-Borel theorem, there is a finite subcover $N$. Let $N = \bigcup_{j=1}^k \{N_{z_j}; z = z_j \in K\}$. Then the family $\mathcal{F}$ is uniformly bounded on $N$ (and hence on $K$). Each member of $\mathcal{F}$ is parabolic on $U$ and, hence, is Hölder continuous on $U$. Since $K$ is compact in $U$, the Hölder coefficients and exponents can be made uniform for the family $\mathcal{F}$ and it follows that $\mathcal{F}$ is an equicontinuous family on $K$. It follows from Arzela’s theorem that each net $\{u_\alpha; \alpha \in A\}$ in $\mathcal{F}$ has a uniformly convergent subnet on $K$.

Let $\{u_\alpha; \alpha \in K\}$ be a convergent net of uniformly bounded parabolic
functions on $U$. Then $u \equiv \lim_{\alpha} u_{\alpha}$ exists. Let $\{K_j\}$ be a sequence of compact sets in $U$ such that $K_j \uparrow U$. It follows from what has been shown that for $K_1$ there is a subnet $\{u_{\alpha}\}_1$ of $\{u_{\alpha}: \alpha \in A\}$ which converges to a parabolic function on $K_1$. Since $u_{\alpha} \rightarrow u$ on $U$, $u$ is parabolic on $K_1$. Using what has been shown on $K_2$, there is a subnet $\{u_{\alpha}\}_2$ of $\{u_{\alpha}\}_1$ which converges to a parabolic function, $u$, on $K_2$. Continuing this process, the diagonalization process can be used to see that $u$ is parabolic on all compact subsets of $U$ and, hence, on $U$ itself.

**Definition 8.** Let $u$ be defined on $U$. Let $U_p = \{z \in U; u$ locally parabolic at $z\}$ and $U_\pm = \{z \in U; u(z) = \pm \infty\}$. $u$ is said to have property $P_\pm$ if

1. $w \in U_p$ implies $z \in U_p$ for all $z \in U$ with $z < w$.
2. $w \in U_\pm$ implies $z \in U_\pm$ for all $z \in U$ with $z > w$.
3. $U = U_p - U_+ - U_-$ is a finite union of sets of the form $\Omega_j \times (t_j)$.

**Theorem 12.** If $u_j$ is an increasing (or decreasing) sequence of parabolic functions on $U$, then $u(z) = \lim u_j(z)$ satisfies property $P_+$ (or $P_-$).

**Proof.** Assume $u_j \uparrow u$. If for some $w \in U$, $u(w) < \infty$, then for any standard rectangle $R$ with $\overline{R} \subset U$ and $w \in R$, the monotone convergence theorem implies that

\[
L(u; w, R) = \lim L(u_j; w, R) = \lim u_j(w) = u(w) < \infty
\]

and, hence, $u(z) < \infty$ on $\partial R$ with $z < w$. Therefore, $U_p$ satisfies property (1) of Definition 8. Since $u \geq u_1 \geq -\infty$ by the parabolicity of $u_1$, $U_- = \emptyset$. To see that $U_+$ satisfies property (2), suppose $u(w) = +\infty$ and suppose for some $z \in U$ with $w < z$ that $u(z) < \infty$. Then by our first argument $u(w) < \infty$ and a contradiction is obtained. Therefore property (2) is satisfied. Since $u_j \uparrow u$, for any $z \in U$ either $u(z) < \infty$ or $u(z) = \infty$ must hold. Hence, property (3) is vacuously satisfied.

**Definition 9.** A family $\mathcal{F}$ of functions defined on $U$ is right-directed if for each pair $u, v \in \mathcal{F}$ there is a function $w \in \mathcal{F}$ such that $u \leq w$ and $v \leq w$. $\mathcal{F}$ is left-directed if the above inequalities are reversed.

**Lemma 3.** If $\{u_i: i \in I\}$ is a right-directed (left-directed) family of functions parabolic on $U$, then $u \equiv \sup \{u_i\}$ $(u \equiv \inf \{u_i\})$ satisfies property $P_+$ (or $P_-$) in $U$.

**Proof.** If $\{u_i: i \in I\}$ contains only one function then we are done. If
\{u_i; i \in I\} contains two or more functions, then the right-directedness can be used to obtain a sequence \{u_j\} in \{u_i; i \in I\} such that \(u_j \uparrow u = \sup_{\tau} u_{\tau}\). The desired result follows from Theorem 12.

**Definition 10.** A family \(\mathcal{F}\) in \(S_U\) is saturated if

1. \(u, v \in \mathcal{F} \Rightarrow u \land v \in \mathcal{F}\),
2. \(u \in \mathcal{F} \Rightarrow u^* \in \mathcal{F}\) where, for some standard rectangle \(R\) in \(U\),

\[
u^*(z) = \begin{cases} u(z) & \text{in } U - R, \\ L(u; z, R) & \text{in } R. \end{cases}
\]

**Theorem 13.** If \(\mathcal{F}\) is a saturated family in \(S_U\), then \(v = \inf_{\mathcal{F}} u\) satisfies property \(P_\_\) in \(U\).

**Proof.** Let \(R\) be an arbitrary standard rectangle. It will be shown that \(v\) satisfies property \(P_\_\) on \(R\) and the desired result will follow from the arbitrariness of \(R\) in \(U\).

For each \(22 \in \mathcal{F}\), let \(\nu^*\) be defined as in (2). By Theorem 10', \(u^* \leq u\) on \(R\). Moreover, since \(\mathcal{F}\) is saturated in \(S_U\), \(u \in \mathcal{F}\) implies \(u^* \in \mathcal{F}\). Therefore,

\[
v = \inf\{u; u \in \mathcal{F}\} = \inf\{u^*; u \in \mathcal{F}\}.
\]

Let \(\mathcal{F}^* = \{u^*; u \in \mathcal{F}\}\). If it can be shown that \(\mathcal{F}^*\) is a left-directed family, the desired result will follow from Lemma 3. Let \(u, w \in \mathcal{F}\). Then \(u \land w \in \mathcal{F}\) and hence \(u^*, w^*,\) and \((u \land w)^* \in \mathcal{F}\). However \(u \land w \leq u\) implies \((u \land w)^* \leq u^*\) and similarly \((u \land w)^* \leq u^*\). That is, for any two elements \(u^*, w^* \in \mathcal{F}^*,\) there is an element, namely \((u \land w)^*\), in \(\mathcal{F}\) such that \((u \land w)^* \leq u^*\) and \((u \land w)^* \leq w^*\). Hence, \(\mathcal{F}^*\) is left-directed and the proof is complete.

The preceding results for families of parabolic and generalized superparabolic functions will next be used to find the greatest parabolic minorant of a given function \(u\) on \(U\). Let \(W\) be an open set in \(U\) and let \(u \in S_U\). Let \(\{R_j\}\) be a sequence of standard rectangles satisfying

1. \(\overline{R_j} \subset W\) for all \(j\),
2. \(W = \bigcup_{j=1}^\infty R_j\),
3. each rectangle \(R_j\) occurs infinitely often in the sequence \(\{R_j\}\).

Define

\[
u_1(z) = \begin{cases} u(z), & z \in U - R_1, \\ L(u; z, R_1), & z \in R_1, \end{cases}
\]

and inductively
\[ u_n(z) = \begin{cases} u_{n-1}(z), & z \in U - R_n, \\ L(u_{n-1}; z, R_n), & z \in R_n. \end{cases} \]

Then by Theorem 10', \( u_n \leq u_{n-1} \) on \( U \), \( u_n \) is parabolic on \( R_n \), and \( u_n \in S_U \). Since \( \{u_n\} \) is a decreasing sequence, \( u_\infty(z) \equiv \lim_{n \to \infty} u_n(z) \) satisfies proper-

**Definition 11.** \( u_\infty \) is called the reduction of \( u \) over \( W \) relative to \( U \).

It appears as if for a given open set \( W \) in \( U \) the corresponding reduction \( u_\infty \) will depend on the defining sequence \( \{R_j\} \). It will follow from Theorem 15 that \( u_\infty \) is independent of the choice of \( \{R_j\} \) if there is a parabolic

**Theorem 14.** If \( u \in S_U \), \( W \) is open in \( U \), and \( u_\infty \) is the reduction of \( u \) over \( W \) relative to \( U \) corresponding to the sequence \( \{R_j\} \), then \( u_\infty \) satisfies property \( P_- \) on each component of \( W \), and \( u_\infty = u \) on \( U - W \).

**Proof.** Without loss of generality assume \( W \) is connected. For each \( j \), there is a sequence \( \{j_k\} \) such that \( R_{j_k} = R_j \) for all \( k \). Thus, on \( R_j \),

\[ u_\infty(z) = \lim_{k \to \infty} u_{j_k}(z). \]

Since \( u_{j_k} \) is parabolic on \( R_j \) for all \( k \), it follows from Theorem 12 that \( u_\infty \) satisfies property \( P_- \) on \( R_j \). Using this procedure on each different rectangle \( R_j \) in the defining sequence, it follows that \( u_\infty \) satisfies property \( P_- \) on \( W \). Since \( u_j = u \) on \( U - W \) for all \( j \), \( u_\infty = u \) on \( U - W \).

**Lemma 4.** If \( u \in S_U \), \( W \) is open in \( U \), \( v \) is parabolic on \( W \), and \( v \leq u \) on \( W \), then \( v \leq u_\infty \leq u \) on \( W \) where \( u_\infty \) is the reduction of \( u \) over \( W \) in \( U \).

**Proof.** Let \( \{R_j\} \) be the sequence of standard rectangles which define the reduction of \( u \) over \( W \), \( u_\infty \). Then

\[ v(z) = L(v; z, R_j) \leq L(u; z, R_j) = u_1(z) \leq u(z) \quad \text{on } R_1, \]

and

\[ v(z) \leq u(z) = u_1(z) \quad \text{on } W - R_1. \]

Therefore, \( v \leq u_1 \leq u \) on \( W \). Proceeding inductively, the desired result is obtained.

**Definition 12.** If \( u \in S_U \), \( v \) is parabolic on \( U \), and \( v \leq u \) on \( U \), then \( v \) is called a parabolic minorant of \( u \). \( v \) is the greatest parabolic minorant of \( u \) if \( v \) is a parabolic minorant of \( u \) and any other parabolic minorant of
Theorem 15. If \( v \in \mathcal{S}_U \), \( W \) is open in \( U \), and \( u \) has a parabolic minorant \( v \) on \( W \), then \( u \) has a unique greatest parabolic minorant on \( W \); namely, \( u_\infty \).

Proof. It follows from Lemma 4 that \( v \leq u_\infty \leq u \) on \( W \). But \( v \) parabolic on \( W \) implies \( v > -\infty \) on \( W \). Therefore, since \( u_\infty \) satisfies property \( P_\infty \) on \( W \), \( u_\infty \) is parabolic on \( W \). If there were two such \( u_\infty \), say \( u_{\infty,1} \) and \( u_{\infty,2} \), then \( u_{\infty,1} \leq u_{\infty,2} \leq u \) on \( W \) since \( u_{\infty,1} \) is a parabolic minorant and \( u_{\infty,2} \leq u_{\infty,1} \leq u \) on \( W \) since \( u_{\infty,2} \) is a parabolic minorant. Therefore, \( u_\infty \) is unique and it is the greatest parabolic minorant of \( u \).

Theorem 16. If \( u, v \in \mathcal{S}_U \) have harmonic minorants, then \( (u + v)_\infty = u_\infty + v_\infty \).

Proof. Since \( u \) and \( v \) have harmonic minorants, \( u + v \) does also and, hence, \( u_\infty, v_\infty \), and \( (u + v)_\infty \) are independent of the defining sequence \( \{R_j\} \). For such a sequence and \( j \geq 1 \)

\[
(u + v)_j(x) = \begin{cases} 
(u + v)_{j-1}(z) & \text{on } U - R_j, \\
L((u + v)_{j-1}; z, R_j) & \text{on } R_j,
\end{cases}
\]

\[
= \begin{cases} 
u_{j-1}(z) & \text{on } U - R_j \\
L(u_{j-1}; z, R_j) & \text{on } R_j
\end{cases}
+ \begin{cases} 
u_{j-1}(z) & \text{on } U - R_j \\
L(v_{j-1}; z, R_j) & \text{on } R_j
\end{cases}
\]

\[
= u_j(z) + v_j(z).
\]

Since \( u_j \downarrow u_\infty \) and \( v_j \downarrow v_\infty \), it follows that \( (u + v)_j \downarrow (u + v)_\infty \) and \( (u + v)_\infty = u_\infty + v_\infty \).

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