CONVERGENT NETS OF PARABOLIC
AND GENERALIZED SUPERPARABOLIC FUNCTIONS

NEIL A. EKLUND

ABSTRACT. The well-known convergence properties of families of harmonic functions are generalized to functions which satisfy $Lu = 0$ where $L$ is the weak parabolic operator in divergence form. Properties of superharmonic functions are obtained for generalized superparabolic functions. These results are obtained on any bounded domain in $E^{n+1}$.

Consider the parabolic operator in divergence form given by

$$Lu = u_t - \{a_{ij}(x, t)u_{x_i} + d_j(x, t)u_{x_j} - b_j(x, t)u, - c(x, t)u.$$ 

In two preceding papers by the author [4], [5], existence, representation, and a maximum principle were obtained for solutions of $Lu = 0$ in the cylindrical domain $Q = \Omega \times (0, T)$ for $\Omega \subset E^n$, and generalized superparabolic functions in $Q$ were introduced. In this article the author will consider convergent nets of parabolic and superparabolic functions on a bounded domain $U$ assumed to be in $E^n \times (0, T)$. Since some of the properties obtained in this article will depend on results for superparabolic functions, it is necessary to restate these results for the domain $U$. The numbering of definitions and theorems in this article continues from those in [5].

Definition 7. Let $z = (x, t), w = (y, s) \in U$. $z < w$ in $U$ if there is a polygonal path $\{C_{z,w}(\alpha), 0 \leq \alpha \leq 1\}$ such that

1. $C_{z,w}(0) = \{z\}, C_{z,w}(1) = \{w\}$,
2. if $C_{z,w}(\alpha) = \{(\xi, \tau)\}$, then $\alpha < \beta$ implies $\tau_\alpha < \tau_\beta$,
3. $\{C_{z,w}(\alpha); 0 \leq \alpha \leq 1\} \subset U$.

Note that Definitions 4, 5, and 6 can be generalized to the domain $U$ since all properties are in terms of standard rectangles in the given domain. Accordingly, let $L_U, E_U', E_U'', E_U'$ denote the corresponding spaces. The
Theorems and corollaries stated below correspond to the analogous results obtained for \( Q \) in [5].

**Theorem 4'.** If \( u \in \mathcal{S}_U \) and if there is a \( z_0 \in U \) such that \( 0 \geq u(z_0) = \inf_U u \), then \( u(z) = u(z_0) \) for all \( z < z_0 \) in \( U \).

The following corollary is the analogue of that preceding Theorem 7 in [5].

**Corollary.** If \( u \in \mathcal{S}_U \), then
1. \( u(z_0) < \infty \) implies \( u(z) < \infty \) for all \( z < z_0 \) in \( U \);
2. \( u(z_0) = \infty \) implies \( u(z) = \infty \) for all \( z < z_0 \) in \( U \).

**Theorem 10'.** Let \( u \in \mathcal{S}_U \) and let \( R \) be a standard rectangle in \( U \). If \( u \not< \infty \) on \( R \), then \( L(u; z, R) \) exists and the function

\[
    v(z) = \begin{cases} 
        u(z), & z \in U - R, \\
        L(u; z, R), & z \in R,
    \end{cases}
\]

satisfies \( u \geq v \), \( L v = 0 \) on \( R \), and \( v \in \mathcal{S}_U \).

**Lemma 2.** If \( \{u_{\alpha}; \alpha \in A\} \) is a net of functions parabolic on \( U \) which converges uniformly on \( U \) to \( u \), then \( u \) is parabolic on \( U \).

**Proof.** For any standard rectangle \( R, \overline{R} \subset U \), and any \( z \in R \), \( u_{\alpha}(z) = L(u_{\alpha}; z, R) \). Since \( u_{\alpha} \rightarrow u \) uniformly on \( R \), it follows that \( u(z) = L(u; z, R) \). Since \( R \) in \( U \) was arbitrary, \( u \) is parabolic on \( U \).

**Theorem 11.** If \( \mathcal{F} \) is a family of parabolic functions on \( U \) uniformly bounded on a neighborhood of each point of \( U \) and if \( K \) is compact in \( U \), then the family \( \mathcal{F} \) is uniformly equicontinuous on \( K \) and each net \( \{u_{\alpha}; \alpha \in A\} \) in \( \mathcal{F} \) has a subnet which converges uniformly on \( K \). If \( \{u_{\alpha}; \alpha \in A\} \) is a convergent net of uniformly bounded parabolic functions on \( U \), then \( u = \lim_A u_{\alpha} \) is parabolic on \( U \).

**Proof.** For each \( z \in K \), let \( N_z \) denote the neighborhood of \( z \) on which the family \( \mathcal{F} \) is uniformly bounded. Then \( \bigcup\{N_z; z \in K\} \) is an open cover for \( K \) and, by the Heine-Borel theorem, there is a finite subcover \( N \). Let \( N = \bigcup_{j=1}^{k} \{N_{z_j}; z = z_j \in K\} \). Then the family \( \mathcal{F} \) is uniformly bounded on \( N \) (and hence on \( K \)). Each member of \( \mathcal{F} \) is parabolic on \( U \) and, hence, is Hölder continuous on \( U \). Since \( K \) is compact in \( U \), the Hölder coefficients and exponents can be made uniform for the family \( \mathcal{F} \) and it follows that \( \mathcal{F} \) is an equicontinuous family on \( K \). It follows from Arzela's theorem that each net \( \{u_{\alpha}; \alpha \in A\} \) in \( \mathcal{F} \) has a uniformly convergent subnet on \( K \).

Let \( \{u_{\alpha}; \alpha \in K\} \) be a convergent net of uniformly bounded parabolic
functions on $U$. Then $u \equiv \lim_{\alpha} u_\alpha$ exists. Let $\{K_\alpha\}$ be a sequence of compact sets in $U$ such that $K_\alpha \uparrow U$. It follows from what has been shown that for $K_1$ there is a subnet $\{u_{\alpha_1}\}_1$ of $\{u_\alpha; \alpha \in A\}$ which converges to a parabolic function on $K_1$. Since $u_\alpha \to u$ on $U$, $u$ is parabolic on $K_1$. Using what has been shown on $K_2$, there is a subnet $\{u_{\alpha_2}\}_2$ of $\{u_{\alpha_1}\}_1$ which converges to a parabolic function, $u$, on $K_2$. Continuing this process, the diagonalization process can be used to see that $u$ is parabolic on all compact subsets of $U$ and, hence, on $U$ itself.

Definition 8. Let $u$ be defined on $U$. Let $U_p = \{z \in U; u$ locally parabolic at $z\}$ and $U_+ = \{z \in U; u(z) = \pm \infty\}$. $u$ is said to have property $P_+$ if

1. $w \in U_p$ implies $z \in U_p$ for all $z \in U$ with $z < w$.
2. $w \in U_+$ implies $z \in U_+$ for all $z \in U$ with $z > w$.
3. $U - U_p - U_+ - U_-$ is a finite union of sets of the form $\Omega_j \times (t_j)$.

Theorem 12. If $u_j$ is an increasing (or decreasing) sequence of parabolic functions on $U$, then $u(z) = \lim u_j(z)$ satisfies property $P_+$ (or $P_-$).

Proof. Assume $u_j \uparrow u$. If for some $w \in U$, $u(w) < \infty$, then for any standard rectangle $R$ with $R \subset U$ and $w \in R$, the monotone convergence theorem implies that

$$L(u_j; w, R) = \lim L(u_j; w, R) = \lim u_j(w) = u(w) < \infty$$

and, hence, $u(z) < \infty$ on $\partial R$ with $z < w$. Therefore, $U_p$ satisfies property (1) of Definition 8. Since $u \geq u_1 \geq -\infty$ by the parabolicity of $u_1$, $U_- = \emptyset$. To see that $U_+$ satisfies property (2), suppose $u(w) = +\infty$ and suppose for some $z \in U$ with $w < z$ that $u(z) < \infty$. Then by our first argument $u(w) < \infty$ and a contradiction is obtained. Therefore property (2) is satisfied. Since $u_j \uparrow u$, for any $z \in U$ either $u(z) < \infty$ or $u(z) = \infty$ must hold. Hence, property (3) is vacuously satisfied.

Definition 9. A family $\mathcal{F}$ of functions defined on $U$ is right-directed if for each pair $u, v \in \mathcal{F}$ there is a function $w \in \mathcal{F}$ such that $u \leq w$ and $v \leq w$. $\mathcal{F}$ is left-directed if the above inequalities are reversed.

Lemma 3. If $\{u_i; i \in I\}$ is a right-directed (left-directed) family of functions parabolic on $U$, then $u \equiv \sup_I u_i$ ($u \equiv \inf_I u_i$) satisfies property $P_+$ ($P_-$) in $U$.

Proof. If $\{u_i; i \in I\}$ contains only one function then we are done. If
\{u_i; \ i \in I\} contains two or more functions, then the right-directedness can be used to obtain a sequence \(\{u_j\}\) in \(\{u_i; \ i \in I\}\) such that \(u_j \uparrow u = \sup_{t} u_t\). The desired result follows from Theorem 12.

**Definition 10.** A family \(\mathcal{F}\) in \(S'_U\) is saturated if

1. \(u, v \in \mathcal{F} \Rightarrow u \land v \in \mathcal{F}\),
2. \(u \in \mathcal{F} \Rightarrow u^* \in \mathcal{F}\) where, for some standard rectangle \(R\) in \(U\),

\[
u^*(z) = \begin{cases} u(z) & \text{in } U - R, \\ L(u; z, R) & \text{in } R. \end{cases}\]

**Theorem 13.** If \(\mathcal{F}\) is a saturated family in \(S'_U\), then \(v = \inf_{\mathcal{F}} u\) satisfies property \(P_-\) in \(U\).

**Proof.** Let \(R\) be an arbitrary standard rectangle. It will be shown that \(v\) satisfies property \(P_-\) on \(R\) and the desired result will follow from the arbitrariness of \(R\) in \(U\).

For each \(u \in \mathcal{F}\), let \(u^*\) be defined as in (2). By Theorem 10', \(u^* \leq u\) on \(R\). Moreover, since \(\mathcal{F}\) is saturated in \(S'_U\), \(u \in \mathcal{F}\) implies \(u^* \in \mathcal{F}\). Therefore,

\[
\nu = \inf_{\mathcal{F}} u = \inf_{\mathcal{F}} u^* = \inf_{\mathcal{F}} u^*.
\]

Let \(\mathcal{F}^* = \{u^*; \ u \in \mathcal{F}\}\). If it can be shown that \(\mathcal{F}^*\) is a left-directed family, the desired result will follow from Lemma 3. Let \(u, w \in \mathcal{F}\). Then \(u \land w \in \mathcal{F}\) and hence \(u^*, w^*, \) and \((u \land w)^* \in \mathcal{F}\). However \(u \land w \leq u\) implies \((u \land w)^* \leq u^*\) and similarly \((u \land w)^* \leq w^*\). That is, for any two elements \(u^*, w^* \in \mathcal{F}^*\), there is an element, namely \((u \land w)^*\), in \(\mathcal{F}\) such that \((u \land w)^* \leq u^*\) and \((u \land w)^* \leq w^*\). Hence, \(\mathcal{F}^*\) is left-directed and the proof is complete.

The preceding results for families of parabolic and generalized superparabolic functions will next be used to find the greatest parabolic minorant of a given function \(u\) on \(U\). Let \(W\) be an open set in \(U\) and let \(u \in \mathcal{S}'_U\). Let \(\{R_j\}\) be a sequence of standard rectangles satisfying

1. \(R_j \subset W\) for all \(j\),
2. \(W = \bigcup_{j=1}^\infty R_j\),
3. each rectangle \(R_j\) occurs infinitely often in the sequence \(\{R_j\}\).

Define

\[
u_1(z) = \begin{cases} u(z) & \text{in } U - R_1, \\ L(u; z, R_1) & \text{in } R_1, \end{cases}\]

and inductively
Nets of Parabolic and Superparabolic Functions

\[ u_n(z) = \begin{cases} 
    u_{n-1}(z), & z \in U - R_n, \\
    L(u_{n-1}; z, R_n), & z \in R_n.
\end{cases} \]

Then by Theorem 10', \( u_n \leq u_{n-1} \) on \( U \), \( u_n \) is parabolic on \( R_n \), and \( u_n \in \mathcal{S}_U^\gamma \). Since \( \{u_n\} \) is a decreasing sequence, \( u_\infty(z) = \lim_{n \to \infty} u_n(z) \) satisfies property \( P_- \) on \( U \).

**Definition 11.** \( u_\infty \) is called the reduction of \( u \) over \( W \) relative to \( U \).

It appears as if for a given open set \( W \) in \( U \) the corresponding reduction \( u_\infty \) will depend on the defining sequence \( \{R_j\} \). It will follow from Theorem 15 that \( u_\infty \) is independent of the choice of \( \{R_j\} \) if there is a parabolic function \( v \) on \( W \) with \( v \leq u \).

**Theorem 14.** If \( u \in \mathcal{S}_U^\gamma \), \( W \) is open in \( U \), and \( u_\infty \) is the reduction of \( u \) over \( W \) relative to \( U \) corresponding to the sequence \( \{R_j\} \), then \( u_\infty \) satisfies property \( P_- \) on each component of \( W \), and \( u_\infty = u \) on \( U - W \).

**Proof.** Without loss of generality assume \( W \) is connected. For each \( j \), there is a sequence \( \{j_k\} \) such that \( R_{j_k} = R_j \) for all \( k \). Thus, on \( R_{j_k} \)

\[ u_{\infty}(z) = \lim_{k \to \infty} u_{j_k}(z). \]

Since \( u_{j_k} \) is parabolic on \( R_j \) for all \( k \), it follows from Theorem 12 that \( u_{\infty} \) satisfies property \( P_- \) on \( R_j \). Using this procedure on each different rectangle \( R_j \) in the defining sequence, it follows that \( u_\infty \) satisfies property \( P_- \) on \( W \). Since \( u_j = u \) on \( U - W \) for all \( j \), \( u_\infty = u \) on \( U - W \).

**Lemma 4.** If \( u \in \mathcal{S}_U^\gamma \), \( W \) is open in \( U \), \( v \) is parabolic on \( W \), and \( v \leq u \) on \( W \), then \( v \leq u_{\infty} \leq u \) on \( W \) where \( u_{\infty} \) is the reduction of \( u \) over \( W \) in \( U \).

**Proof.** Let \( \{R_j\} \) be the sequence of standard rectangles which define the reduction of \( u \) over \( W \), \( u_\infty \). Then

\[ v(z) = L(v; z, R_1) \leq L(u; z, R_1) = u_1(z) \leq u(z) \quad \text{on } R_1, \]

and

\[ v(z) \leq u(z) = u_1(z) \quad \text{on } W - R_1. \]

Therefore, \( v \leq u_1 \leq u \) on \( W \). Proceeding inductively, the desired result is obtained.

**Definition 12.** If \( u \in \mathcal{S}_U^\gamma \), \( v \) is parabolic on \( U \), and \( v \leq u \) on \( U \), then \( v \) is called a parabolic minorant of \( u \). \( v \) is the greatest parabolic minorant of \( u \) if \( v \) is a parabolic minorant of \( u \) and any other parabolic minorant of
Theorem 15. If \( v \in \mathcal{S}_U \), \( W \) is open in \( U \), and \( u \) has a parabolic minorant \( v \) on \( W \), then \( u \) has a unique greatest parabolic minorant on \( W \); namely, \( u_\infty \).

Proof. It follows from Lemma 4 that \( v \leq u_\infty \leq u \) on \( W \). But \( v \) parabolic on \( W \) implies \( v > -\infty \) on \( W \). Therefore, since \( u_\infty \) satisfies property \( P \) on \( W \), \( u_\infty \) is parabolic on \( W \). If there were two such \( u_\infty \), say \( u_{\infty,1} \) and \( u_{\infty,2} \), then \( u_{\infty,1} \leq u_{\infty,2} \leq u \) on \( W \) since \( u_{\infty,1} \) is a parabolic minorant and \( u_{\infty,2} \leq u_{\infty,1} \leq u \) on \( W \) since \( u_{\infty,2} \) is a parabolic minorant. Therefore, \( u_\infty \) is unique and it is the greatest parabolic minorant of \( u \).

Theorem 16. If \( u, v \in \mathcal{S}_U \) have harmonic minorants, then \( (u + v)_\infty = u_\infty + v_\infty \).

Proof. Since \( u \) and \( v \) have harmonic minorants, \( u + v \) does also and, hence, \( u_\infty, v_\infty \), and \( (u + v)_\infty \) are independent of the defining sequence \( \{R_j\} \). For such a sequence and \( j \geq 1 \)

\[
(u + v)_j(z) = \begin{cases} 
(u + v)_{j-1}(z) & \text{on } U - R_j, \\
L((u + v)_{j-1}; z, R_j) & \text{on } R_j,
\end{cases}
\]

\[
= \begin{cases} 
(u_{j-1}(z) & \text{on } U - R_j, \\
L(u_{j-1}; z, R_j) & \text{on } R_j
\end{cases} + \begin{cases} 
v_{j-1}(z) & \text{on } U - R_j, \\
v_{j-1}(z) & \text{on } R_j
\end{cases}
\]

\[
= u_j(z) + v_j(z).
\]

Since \( u_j \downarrow u_\infty \) and \( v_j \downarrow v_\infty \), it follows that \( (u + v)_j \downarrow (u + v)_\infty \) and \( (u + v)_\infty = u_\infty + v_\infty \).

REFERENCES


DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37235

Current address: Department of Mathematics, Centre College, Danville, Kentucky 40422