ABSTRACT. E. F. Steiner introduced a quasi-proximity $\delta$ satisfying $A \delta B$ iff $\{x\} \delta B$ for some $x$ of $A$. The purpose of this paper is to describe the Tychonoff product of topologies in terms of Steiner's quasi-proximities. Whenever $(X_a, \delta_a)$ is the Steiner quasi-proximity space, the product proximity on $X = \prod_{a} X_a$ can be given, by using the concept of finite coverings, as the smallest proximity on $X$ which makes each projection $\delta$-continuous.

Introduction. E. F. Steiner [2] introduced a quasi-proximity $\delta$ satisfying $A \delta B$ iff $\{a\} \delta B$ for some $a$ of $A$. This note is devoted to the study of a product proximity on $X = \prod_{a} X_a$, where each $(X_a, \delta_a)$ is the above Steiner quasi-proximity space. As F. W. Stevenson [3] pointed out, there are three equivalent definitions of a product proximity. Especially, Császár and Leader defined a product proximity by using finite coverings [3]. Unfortunately, for Steiner's quasi-proximity, it seems difficult to us to define the product proximity in the same way as Császár and Leader. We must modify the definition of a product proximity in our case (Definition 2). We then show that the Tychonoff product topology can be induced on the cartesian product $X = \prod_{a} X_a$ in terms of the quasi-proximity mentioned above.

The reader is referred to S. A. Naimpally and B. D. Warrack [1] for definitions not given here.

Preliminary definitions and lemmas.

Definition 1. A binary relation $\delta$ defined on the power set of $X$ is called a Steiner's or $S$-quasi-proximity on $X$ iff $\delta$ satisfies the axioms below.

(I) For every $A \subset X$, $A \overline{\delta} \phi$ ($\overline{\delta}$ means "not-$\delta$").

(II) $A \delta B$ iff $\{a\} \delta B$ for some $a \in A$.

(III) $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$.

(IV) For every $x \in X$, $\{x\} \delta \{x\}$.

(V) $A \overline{\delta} B$ implies that there exists a subset $C$ such that $A \overline{\delta} C$ and $(X - C) \overline{\delta} B$. 
Remark 1. Clearly Axiom (II) is equivalent to Axiom (II') below.

(II') For an arbitrary index set $\Lambda$,

$$\left( \bigcup_{\lambda \in \Lambda} A_{\lambda} \right) \delta B \iff A_{\mu} \delta B \text{ for some } \mu \in \Lambda.$$

Furthermore, in the $S$-quasi-proximity we can replace Axiom (V) with Axiom (V') below.

(V') If $x \ll A$, then there exists a set $B$ such that $x \ll B \ll A$. (In general, $P \ll Q$ means $P \overline{\delta} (X - Q)$ and $Q$ is said to be a $\delta$-neighborhood of $P$.)

In fact, it is easily seen that Axiom (V) implies Axiom (V'). Conversely we show that Axiom (V) follows from Axioms (I)–(IV) and (V'). Suppose $A \overline{\delta} B$. By Axiom (II), $\{x\} \overline{\delta} B$, i.e. $x \ll X - B$ for each $x \in A$. Then it follows from Axiom (V') that there is a set $C_x$ such that $x \ll C_x \ll X - B$ for each $x \in A$. Since $\{x\} \overline{\delta} (X - C_x)$ for each $x \in A$,

$$\{x\} \overline{\delta} \left( X - \bigcup_{x \in A} C_x \right)$$

by Axiom (III).

Setting $\bigcup_{x \in A} C_x = C$, we obtain $A \overline{\delta} (X - C)$ by Axiom (II). On the other hand, since $C_x \overline{\delta} B$ for each $x \in A$, we have $C \overline{\delta} B$ by Axiom (II'). Thus Axiom (V) surely holds.

Let $(X, \delta)$ be an $S$-quasi-proximity space. For every $A \subseteq X$, we set $c(A) = \{x : \{x\} \delta A\}$. Then the operator $c$ is a topological closure operator and so $X$ is a topological space [2]. This topological space is denoted by $(X, c)$ and the topology induced by $\delta$ is denoted by $\tau(\delta)$. If, on a set $X$, there is a topology $\tau$ and a proximity $\delta$ such that $\tau = \tau(\delta)$, then $\tau$ and $\delta$ are said to be compatible.

The proof of the following is trivial.

Lemma 1. (1) If $A \delta B$ and $B \subseteq C$, then $A \delta C$.
(2) If $A \delta B$ and $A \subseteq C$, then $C \delta B$.
(3) If $A \overline{\delta} B$, then $A \cap B = \emptyset$.

Lemma 2. For subsets $A$ and $B$ of an $S$-quasi-proximity space $(X, c)$,

$$A \delta B \iff A \cap c(B) \neq \emptyset \iff A \delta c(B).$$

Proof. This follows readily from Axiom (II).

The following is a direct result of Lemma 2.

Lemma 3. Every topological space $(X, \tau)$ with the topology $\tau$ has a
compatible $S$-quasi-proximity $\delta$ defined by

$$A \delta B \iff A \cap \overline{B} \neq \emptyset,$$

where $\overline{B}$ denotes the $\tau$-closure of $B$.

The following lemma shows that in $S$-quasi-proximity spaces a $\delta$-continuous mapping and a continuous mapping are equivalent.

**Lemma 4.** Let $f$ be a mapping of an $S$-quasi-proximity space $(X, \delta_1)$ into an $S$-quasi-proximity space $(Y, \delta_2)$. Then $f$ is $\delta$-continuous if and only if it is a continuous mapping of the topological space $(X, \tau(\delta_1))$ into the topological space $(Y, \tau(\delta_2))$.

**Proof.** Suppose that $f$ is $\delta$-continuous and that $x$ is any point of $c_1(A)$. Then $\{x\} \delta_1 A$, which implies $f(\{x\}) \delta_2 f(A)$. It follows that $f(\{x\}) \in c_2\{f(A)\}$ and so $f(c_1(A)) \subseteq c_2\{f(A)\}$. ($c_1$ and $c_2$ denote the closure operators in $(X, \delta_1)$ and $(Y, \delta_2)$ respectively.) Conversely let $f$ be continuous and let $A \delta_1 B$. Since, by Lemma 2 $A \cap c_1(B) \neq \emptyset$, it follows that $f(A) \cap c_2\{f(B)\} \neq \emptyset$. From the continuity of $f$, we obtain that $f(A) \cap c_2\{f(B)\} \neq \emptyset$. This implies $f(A) \delta_2 f(B)$, so that $f$ is $\delta$-continuous. Q. E. D.

**Proximity products.** In the present section we attempt to obtain a direct construction of an $S$-quasi-proximity product space by a proximal approach. As we stated in the introduction, we modify the definition of Császár and Leader for the product proximity.

**Definition 2.** Let \{$(X_a, \delta_a) : a \in \Lambda$\} be an arbitrary family of $S$-quasi-proximity spaces. Let $X = \prod_{a \in \Lambda} X_a$ denote the cartesian product of these spaces. A binary relation $\delta$ on the power set of $X$ is defined as follows:

Let $A$ and $B$ be subsets of $X$. Define $A \delta B$ iff there is a point $x_0 \in A$ such that, for any finite covering \{$B_i : i = 1, 2, \ldots, n$\} of $B$, there exists a set $B_i$ satisfying $P_a[x_0] \delta_a P_a[B_i]$ for each $a \in \Lambda$, where each $P_a$ denotes the projection from $X$ to $X_a$.

**Remark 2.** Leader [3] defined a product proximity as follows: For $A$, $B \subseteq X$, $A \delta B$ iff for any finite coverings \{$A_i : i = 1, 2, \ldots, m$\} and \{$B_j : j = 1, 2, \ldots, n$\} of $A$ and $B$ respectively, there is an $A_i$ and a $B_j$ such that $P_a[A_i] \delta_a P_a[B_j]$ for each $a \in \Lambda$. But in order to prove that $\delta$ satisfies Axiom (II), it seems difficult to use Leader's definition for the $S$-quasi-proximity.

**Lemma 5.** Let each $(X_a, \delta_a)$ be an $S$-quasi-proximity space and let $A$
and $B$ be subsets of $X = \prod X_a$. Then $A \triangleleft B$ implies $P_a[A] \delta_a P_a[B]$ for each $a \in \Lambda$.

**Proof.** Suppose $A \triangleleft B$. Since $\{B\}$ itself is a finite covering of $B$, there is a point $x_0$ of $A$ such that $P_a[x_0] \delta_a P_a[B]$ for each $a \in \Lambda$. Applying Axiom (II) to each $\delta_a$, we have $P_a[A] \delta_a P_a[B]$ for each $a \in \Lambda$. Q. E. D.

It follows from Lemma 5 that each projection $P_a$ is $\delta$-continuous and hence it is also continuous by Lemma 4 if $X$ becomes an $S$-quasi-proximity space. Now we prove the main theorem.

**Theorem 1.** The binary relation $\delta$ given by Definition 2 is an $S$-quasi-proximity on the cartesian product $X$. This space $(X, \delta)$ is said to be an $S$-quasi-proximity product space.

**Proof.** It suffices to show that $\delta$ satisfies Axioms (I)–(IV) of Definition 1 and Axiom (V) of Remark 1. It is easy to see that $\delta$ satisfies Axiom (I).

Axiom (II): Suppose $A \triangleleft B$. If $x_0 \in A$ fulfils the condition in Definition 2, then clearly $x_0 \triangleleft B$.

Conversely suppose that $\{x_0\} \triangleleft B$ for some $x_0$ of $A$. If $\{B_i: i = 1, 2, \ldots, n\}$ is any finite covering of $B$, then there is a set $B_i$ such that $P_a[x_0] \delta_a P_a[B_i]$ for each $a \in \Lambda$. By Definition 2, this means $A \triangleleft B$.

Axiom (III): Suppose $A \triangleleft B$ and let $x_0 \in A$ satisfy the condition in Definition 2. If $\{D_i: i = 1, 2, \ldots, n\}$ is any finite covering of $B \cup C$, then it is a covering of $B$ as well; hence there is an $i$ such that $P_a[x_0] \delta_a P_a[D_i]$ for each $a \in \Lambda$. Thus $A \triangleleft (B \cup C)$.

Conversely suppose $A \triangledown B$ and $A \triangledown C$. Then for any given $x \in A$, there are finite coverings $\{D_i: i = 1, 2, \ldots, n\}$ and $\{D_j: j = n + 1, \ldots, n + p\}$ of $B$ and $C$ respectively such that

$$P_a[x] \triangledown_a P_a[D_i] \text{ for } a = t \in \Lambda,$$

$$P_a[x] \triangledown_a P_a[D_j] \text{ for } a = s \in \Lambda,$$

where $i = 1, 2, \ldots, n$ and $j = n + 1, \ldots, n + p$. Since $\{D_k: k = 1, 2, \ldots, n + p\}$ is a covering of $B \cup C$, we conclude that $A \triangledown (B \cup C)$.

Axiom (IV): Let $x$ be a point of $X$ and let $A$ be any set such that $x \in A$. Since $P_a[x] \in P_a[A]$ for each $a \in \Lambda$, by Lemma 1(3) we have $P_a[x] \delta_a P_a[A]$ for each $a \in \Lambda$. Thus $\{x\} \delta \{x\}$.

Axiom (V): Let $\{x\}$ and $A$ be subsets of $X$ such that $x \ll A$, that is, $\{x\} \triangledown (X - A)$. Then there is a finite covering $\{A_i: i = 1, 2, \ldots, n\}$ of $(X - A)$ such that $P_a[x] \triangledown_a P_a[A_i]$ for some $a_i \in \Lambda$, where $i = 1, 2, \ldots, n$. 

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Equivalently \( P_a[x] \ll X_a - P_a[A_i] \). Since each \( \delta_a \) satisfies Axiom (V'), there exist \( G_i \) \((i = 1, 2, \ldots, n)\) such that

\[
(1) \quad P_a[x] \ll G_i \ll X_a - P_a[A_i] \quad \text{for } a = t_i \in \Lambda.
\]

From the first half of (1), we have

\[
(2) \quad P_a[x] \overline{\delta}_a (X_a - G_i).
\]

Now we set

\[
K_i = P_a^{-1}[X_a - G_i] = X - P_a^{-1}[G_i]
\]

and set \( K = \bigcup_{i=1}^n K_i \). It follows from (2) that

\[
P_a[x] \overline{\delta}_a P_a[K_i] \quad \text{for } a = t_i \in \Lambda, \quad i = 1, 2, \ldots, n.
\]

Since \( \{K_i: i = 1, 2, \ldots, n\} \) is a finite covering of \( K \), we obtain \( \{x\} \overline{\delta} K \). This implies

\[
(3) \quad x \ll X - K.
\]

Next, from the second half of (1), we have

\[
(4) \quad G_i \overline{\delta}_a P_a[A_i] \quad \text{for some } a = t_i, \quad i = 1, 2, \ldots, n.
\]

On the other hand, since

\[
X - K = \bigcap_{j=1}^n P_a^{-1}[G_j] \quad (a = t_i),
\]

it follows that

\[
P_a[X - K] = P_a \left\{ \bigcap_{j=1}^n P_a^{-1}[G_j] \right\} \subseteq G_i \quad \text{for } a = t_i.
\]

Hence for every point \( y \) of \( X - K \),

\[
P_a[y] \in G_i \quad (a = t_i; \quad i = 1, 2, \ldots, n).
\]

By (4) and Lemma 1(2), we have therefore \( P_a[y] \overline{\delta}_a P_a[A_i] \) for every \( y \) of \( X - K \), where \( a = t_i; \quad i = 1, 2, \ldots, n \). Because \( \{A_i: i = 1, 2, \ldots, n\} \) is a finite covering of \( (X - A) \), we get that

\[
(5) \quad (X - K) \overline{\delta} (X - A), \quad \text{that is, } \quad X - K \ll A.
\]

Relations (3) and (5) together show that \( \overline{\delta} \) satisfies Axiom (V'). This completes the proof.
In view of Lemma 4, the following theorem shows that the Tychonoff product topology can be induced on an S-quasi-proximity product space \((X, \pi(\delta))\).

**Theorem 2.** The S-quasi-proximity \(\delta\) on \(X\) given by Definition 2 is the smallest S-quasi-proximity for which each projection \(P_a\) is \(\delta\)-continuous.

**Proof.** Let \(\beta\) be an arbitrary S-quasi-proximity on \(X\) such that each projection \(P_a\) is a \(\delta\)-continuous mapping of \((X, \beta)\) into \((X_a, \delta_a)\). Then we must show that \(A \beta B\) implies \(A \delta B\) for \(A, B \subset X\). By Axiom (II), there is a point \(x_0\) of \(A\) such that \(\{x_0\} \beta B\). Given any finite covering \(\{B_i: i = 1, 2, \ldots, n\}\) of \(B\), we can choose a set \(B_i\) such that \(\{x_0\} \beta B_i\) by Axiom (III). Since each \(P_a\) is \(\delta\)-continuous, \(P_a[x_0] \delta_a P_a[B_i]\) for each \(a \in \Lambda\). Because of Definition 2, we can conclude \(A \delta B\). Q. E. D.

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**REFERENCES**

