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AXIOM OF CHOICE AND COMPLEMENTATION

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ABSTRACT. It is shown that an intuitionistic model of set theory with the axiom of choice has to be a classical one.

A topos $\mathcal{E}$ is a category which has *finite limits* (i.e. finite products, intersections and a terminal object, 1), a *universal monomorphism* $1 \xrightarrow{\text{true}} \Omega$ (i.e. for any monomorphism of $\mathcal{E}$ $A' \xrightarrow{m} A$ there exists a unique "characteristic function" such that the diagram

$$\begin{array}{ccc} A' & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ A & \underset{\chi_m}{\longrightarrow} & \Omega \end{array}$$

is a pull-back), and for each object its power set Ω^A (this is characterized by the fact that the morphisms $X \longrightarrow \Omega^A$ are precisely the subobjects of $X \times A$, in particular its global sections $1 \longrightarrow \Omega^A$ are the subobjects of A). The most common examples of topos are the category of sets, $\mathcal{S}$, categories of functors $\mathcal{S}(\mathbf{C}^{\mathrm{op}})$ for any small category $\mathbf{C}$, and categories of sheaves on topological spaces. Details about these can be found in [1] or [2]. One of the main consequences of the axioms is that Ω (the "truth table" object) is a Heyting algebra object. (A Heyting algebra is a lattice with "pseudocomplements". The open set lattice of a topological space is a typical example.) Roughly speaking a topos could be thought of as a model for intuitionistic set theory (subobjects do not have honest complements).

In this setting the axiom of choice reads:

AC: Every epimorphism has a section.

Theorem. *Any coequalizer of two nonintersecting monomorphisms has a section iff in $\mathcal{E}$ subobjects have complements.*

Proof. Let $A' \xrightarrow{m} A$ be a monomorphism in $\mathcal{E}$ and construct the fol-

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lowing coequalizer diagram:

$$(1) \qquad A' \underset{mi_2}{\overset{mi_1}{\rightrightarrows}} A + A \xrightarrow{p} \frac{A+A}{A'}$$

in which by hypothesis p has a splitting. $\frac{A+A}{A'}$ can also be obtained from the push-out diagram:

$$(2) \qquad \begin{array}{ccc} A' & \xrightarrow{m} & A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle j_1} \\ A & \xrightarrow[j_2]{} & \frac{A+A}{A'} \end{array}$$

From the general theory of topos it follows that j_1 and j_2 are monomorphisms and that (2) is a pull-back (hence an intersection).

Let $\frac{A+A}{A'} \xrightarrow{s} A + A$ be a section of p. The mere existence of such a morphism (see [1] or [2]) forces $\frac{A+A}{A'}$ to be the form $D + E$ where $D = s^{-1}(A + 0)$, $E = s^{-1}(0 + A)$. Similarly $s \cdot j_1$ and $s \cdot j_2$ produce two decompositions of A, namely $A = B_1 + C_1$ and $A = B_2 + C_2$, hence

$$A = B_1 \cap B_2 + B_1 \cap C_2 + C_1 \cap B_2 + C_1 \cap C_2.$$

Thus the diagram (2) becomes

$$(3) \qquad \begin{array}{ccc} A' & \xrightarrow{m} & B_1 + C_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle k_1 + l_1} \\ B_2 + C_2 & \xrightarrow[k_2 + l_2]{} & D + E \end{array}$$

therefore $A' = B_1 \cap B_2 + C_1 \cap C_2$ and obviously has a complement $\neg A' = B_1 \cap C_2 + C_1 \cap B_2$.

Conversely, if

$$A' \underset{n}{\overset{m}{\rightrightarrows}} A$$

are such that $m \cap n = 0$ then $A = m + \neg m$ and $A = n + \neg n$, hence

$$A = m \cap \neg n + \neg m \cap n + \neg m \cap \neg n = m + n + \neg m \cap \neg n$$

i.e. $A = A' + A' + B$ for some object B. But then the coequalizer of m and n is

$$A' \underset{i_2}{\overset{i_1}{\rightrightarrows}} A' + A' + B \xrightarrow{\nabla + B} A' + B$$

which is obviously split by

$$A' + B \xrightarrow{i_1 + B} A' + A' + B.$$

Corollary. *AC implies that every subobject has a complement.*

Corollary. *If in* Sh(T) *epimorphisms* (*or even only coequalizers of nonintersecting monomorphisms*) *split then every open set in* T *is clopen and* T *is the disjoint union of sets with the indiscrete topology.*

The present version of the proof is the result of several discussions with M. Barr in which he pointed out that the amount of topos language can be reduced to a minimum.

REFERENCES

1. F. W. Lawvere (editor), *Toposes*, Algebraic Geometry and Logic, Lecture Notes in Math., vol. 274, Springer-Verlag, Berlin and New York, 1972.

2. M. Tierney, *Axiomatic sheaf theory*, Some Constructions and Applications in Categories and Commutative Algebra (P. Salmon, editor), Edizioni Cremonese, Roma, 1973.

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