HYPONORMAL OPERATORS WITH INFINITE ESSENTIAL SPECTRUM

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ABSTRACT. It is shown that if \( T \) is an essentially hyponormal operator (i.e., the image of \( T^*T - TT^* \) in the Calkin algebra is a positive element) in \( \mathcal{L}(\mathcal{H}) \), and if the left essential spectrum of \( T \) is infinite, then \( R(\delta_T)^- + \{ T^* \}' \) is not norm dense in \( \mathcal{L}(\mathcal{H}) \), where \( R(\delta_T)^- \) denotes the norm closure of the range of derivation induced by \( T \), and \( \{ T^* \}' \) denotes the commutant of \( T^* \).

Introduction. Let \( \mathcal{H} \) be a separable, infinite dimensional, complex Hilbert space, and let \( \delta_A(x) = AX - XA \) be an inner derivation on the algebra \( \mathcal{L}(\mathcal{H}) \) of all bounded, linear operators on \( \mathcal{H} \). The range of \( \delta_A \) will be denoted by \( R(\delta_A) \), the closure of \( R(\delta_A) \) in the norm topology will be denoted by \( R(\delta_A)^- \), and the commutant of \( A \) will be denoted by \( \{ A \}' \). Throughout this paper, the term operator will always mean a bounded operator in \( \mathcal{L}(\mathcal{H}) \), \( \Pi_0(A) \) will denote the point spectrum (i.e., the set of eigenvalues) of \( A \), and \( \mathbb{C} \) will denote the complex field. In what follows, the spectrum of an operator \( A \) will be denoted by \( \sigma(A) \), and the left essential spectrum will be denoted by \( E_L(A) \): we note that \( E_L(A) \) is the set of all complex numbers \( \lambda \) such that the image of \( A - \lambda \) in the Calkin algebra is not left invertible. If \( C_2 \) denotes the Hilbert space of Hilbert-Schmidt operators in \( \mathcal{L}(\mathcal{H}) \), then it is known \([8, p. 38]\) that every operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) induces a direct sum decomposition \( C_2 = R(\delta_A)^- \oplus \{ A^* \}' \cap C_2 \), where \( R(\delta_A)^- \) denotes the closure of \( \{ AX - XA : X \in C_2 \} \) in the Hilbert-Schmidt norm. The fact that \( R(\delta_A)^- \) is the orthogonal complement of \( \{ A^* \}' \cap C_2 \) in \( C_2 \) for every operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) indicates that the norm closure of \( R(\delta_A)^- + \{ A^* \}' \) is a natural subspace of \( \mathcal{L}(\mathcal{H}) \), which is associated with each operator \( A \) in \( \mathcal{L}(\mathcal{H}) \).

Recently J. Anderson [1] has shown that if \( N \) is a normal operator, then...
$R(\delta_N)^{-} + \{N^*\}'$ is norm dense in $\mathcal{L}(\mathcal{H})$ if and only if $\sigma(N)$ is finite. However, $R(\delta_{N+K})^{-}$ is not, in general, complemented by $\{N^* + K^*\}'$ in $\mathcal{L}(\mathcal{H})$, even if the spectrum of $N$ is finite: a simple counterexample is given at the end of Corollary 4.

The purpose of this paper is to continue the investigation of when $R(\delta_N)^{-} + \{N\}'$ is norm dense in $\mathcal{L}(\mathcal{H})$. The main result of this paper shows that if $D$ is a diagonalizable normal operator in $\mathcal{L}(\mathcal{H})$ of uniform infinite multiplicity (i.e., each eigenvalue of $D$ has infinite multiplicity), and if $\Pi_0(D)$ is infinite, then for every compact operator $K$, $R(\delta_{D+K})^{-} + \{D^* + K^*\}'$ is not norm dense in $\mathcal{L}(\mathcal{H})$. We also show, as a corollary, that if $T$ is an essentially hyponormal operator (i.e., the image of $T^*T - TT^*$ in the Calkin algebra is a positive element), and if $E_I(T)$ is infinite, then $R(\delta_T)^{-} + \{T^*\}'$ is not norm dense in $\mathcal{L}(\mathcal{H})$.

We begin with our main theorem.

**Theorem 1.** If $D$ is a diagonalizable normal operator in $\mathcal{L}(\mathcal{H})$ of uniform infinite multiplicity, and if $\Pi_0(D)$ is infinite, then for every compact operator $K$, $R(\delta_{D+K})^{-} + \{D^* + K^*\}'$ is not norm dense in $\mathcal{L}(\mathcal{H})$.

**Proof.** Let $\{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of distinct eigenvalues of $D$, which converges to a fixed point $\lambda_0$, and let $K$ be a fixed compact operator in $\mathcal{L}(\mathcal{H})$. By assumption, the subspace $\mathcal{E}_n = \{x \in \mathcal{H} : Dx = \lambda_n x\}$ is infinite dimensional for every positive integer $n$. Since $\|Kx_n\| \to 0$ for every infinite sequence of orthonormal vectors $\{x_n\}$ in $\mathcal{H}$, and since $\mathcal{E}_n$ is infinite dimensional for every $n$, there exists an eigenvector $e_n$ in $\mathcal{E}_n$ such that $De_n = \lambda_n e_n$, $\|e_n\| = 1$, and $\|K^*e_n\| \leq |\lambda_n - \lambda_{n+1}|/2n$ for each positive integer $n$.

Similarly, there exists an eigenvector $f_n$ in $\mathcal{E}_{n+1}$ such that $Df_n = \lambda_{n+1}f_n$, $\|f_n\| = 1$, and $\|K^*f_n\| \leq |\lambda_n - \lambda_{n+1}|/2n$ for every $n$. By the way $e_n$'s and $f_n$'s are defined, it is obvious that $(\|K^*e_n\| + \|K^*f_n\|)/|\lambda_n - \lambda_{n+1}| \leq 1/n$ for every positive integer $n$.

Hence

\[
\lim_{n \to \infty} \frac{\|K^*e_n\| + \|K^*f_n\|}{|\lambda_n - \lambda_{n+1}|} = 0.
\]

Let $U$ be a partial isometry so that $U e_n = f_n$ for every positive integer $n$, and let $U = 0$ on the orthogonal complement of $M = \bigvee_{n=1}^\infty e_n$: for our purpose, it is sufficient to let $U$ be any bounded operator on $M^\perp$. We will now show that

\[
|a| = \|(D + K)X - X(D + K) + S - U\| \geq 1
\]
for each fixed $X$ in $\mathcal{L}(\mathcal{H})$ and $S$ in $\{D^* + K^*\}'$.

A simple calculation shows that

$$
\beta_n = (((D + K)X - X(D + K))e_n, f_n) = ((DX - XD)e_n, f_n) + ((KX - XK)e_n, f_n)
$$

$$
= (\lambda_{n+1} - \lambda_n)(Xe_n, f_n) + ((KX - XK)e_n, f_n).
$$

Since $|\lambda_{n+1} - \lambda_n| \to 0$ and $KX - XK$ is a compact operator,

$$
\lim_{n \to \infty} |\beta_n| = 0.
$$

Using the fact that $S$ commutes with $D^* + K^*$, we have

$$
(S(D^* + K^*))e_n, f_n = \overline{\lambda}_n (Se_n, f_n) + (SK^*e_n, f_n) = ((D^* + K^*)Se_n, f_n)
$$

$$
= (Se_n, \lambda_{n+1}f_n) + (K^*Se_n, f_n) = \overline{\lambda}_{n+1}(Se_n, f_n) + (K^*Se_n, f_n).
$$

Therefore,

$$
(\overline{\lambda}_n - \overline{\lambda}_{n+1})(Se_n, f_n) = ((K^*S - SK^*)e_n, f_n)
$$

and

$$
|((K^*S - SK^*)e_n, f_n)| \leq \frac{|(K^*S - SK^*)e_n, f_n|}{|\lambda_n - \lambda_{n+1}|} \leq \|S\| \frac{\|K^*e_n\| + \|Kf_n\|}{|\lambda_n - \lambda_{n+1}|}.
$$

By (1), it follows that $\lim_{n \to \infty} |(Se_n, f_n)| = 0$. Since $\alpha \geq |\beta_n + (Se_n, f_n) - (Ue_n, f_n)|$ for all positive integers $n$, and since

$$
\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} (Se_n, f_n) = 0,
$$

it is obvious that $\alpha = \|(D + K)X - X(D + K) + S - U\| \geq 1$ for every $X$ in $\mathcal{L}(\mathcal{H})$ and $S$ in $\{D^* + T^*\}'$.

Therefore, $R(\delta_{D+K})^- + \{D^* + K^*\}'$ is not norm dense in $\mathcal{L}(\mathcal{H})$ for every compact operator $K$.

If, in the proof of Theorem 1, we considered an operator $S$ in $\{D + K\}'$, instead of $S$ in $\{D^* + K^*\}'$, the results of (1) and (2) would still be valid. Therefore, under the assumption of Theorem 1, $R(\delta_{D+K})^- + \{D + K\}'$ is also not dense in $\mathcal{L}(\mathcal{H})$, in the norm topology, for every compact operator $K$.

In the following, we derive several corollaries of our main theorem.

**Corollary 2.** Let $T = D \oplus T_1$ be an operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. If $D$ is a diagonable normal operator in $\mathcal{L}(\mathcal{H})$ of uniform infinite multiplicity, and if $\Pi_0(D)$ is infinite, then for every compact operator $K$ in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, $R(\delta_{T+K})^- + \{T^* + K^*\}'$ is not norm dense in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

**Proof.** Let $K$ be a fixed compact operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, and let

$$
\{\lambda_n : n = 1, 2, \ldots\}
$$

be a sequence of distinct eigenvalues of $D$ which converges to a point $\lambda_0$. By Theorem 1, there exists a sequence of orthonormal
eigenvectors \( \{e_n, f_n : n = 1, 2, \ldots \} \) such that
\[
D e_n = \lambda_n e_n, \quad D f_n = \lambda_{n+1} f_n \quad \text{and} \quad \frac{||K^* e_n|| + ||K f_n||}{|\lambda_n - \lambda_{n+1}|} \leq \frac{1}{n}
\]
for every positive integer \( n \). If we let \( U \) be the partial isometry that maps each \( e_n \) to \( f_n, n = 1, 2, \ldots \), and let \( U = 0 \) on the orthogonal complement of \( \bigcup_{n=1}^{\infty} e_n \), then the rest of the proof follows exactly as in the case of Theorem 1.

The following definition and notations are needed in the sequel. Let \( \pi \) denote the canonical quotient map from \( \mathcal{L}(\mathcal{H}) \) onto the Calkin algebra \( \mathcal{L}(\mathcal{H})/\mathcal{K} \), where \( \mathcal{K} \) denotes the closed ideal of all compact operators in \( \mathcal{L}(\mathcal{H}) \).

Definition. A complex number \( \lambda \) is a reducing essential eigenvalue of \( T \) in \( \mathcal{L}(\mathcal{K}) \) if there exists a projection \( P \) in \( \mathcal{L}(\mathcal{H}) \) such that \( \pi(P) \neq 0 \) and \( \pi(T - \lambda)P = \pi(T^* - \overline{\lambda})P = 0 \). The set of all reducing essential eigenvalues of an operator \( T \) will be denoted by \( R_e(T) \).

Corollary 3. Let \( T \) be an operator in \( \mathcal{L}(\mathcal{K}) \). If \( R_e(T) \) is infinite, then \( R(\delta_T)^- + \{T^*\}' \) is not norm dense.

Proof. If \( R_e(T) \) is infinite, then, by Theorem 4.5 in [6, p. 570], \( T \) is unitarily equivalent to an operator \( (D \oplus T_1) + K \) in \( \mathcal{L}(\mathcal{K} \oplus \mathcal{K}) \), where \( D \) is a diagonable normal operator in \( \mathcal{L}(\mathcal{K}) \) of uniform infinite multiplicity, \( \Pi_0(D) \) is infinite, \( T_1 \in \mathcal{L}(\mathcal{K}) \), and \( K \) is a compact operator in \( \mathcal{L}(\mathcal{K} \oplus \mathcal{K}) \). Therefore, by Corollary 2, \( R(\delta_T)^- + \{T^*\}' \) is not norm dense in \( \mathcal{L}(\mathcal{K}) \).

Corollary 4. If \( T \) is an essentially hyponormal operator (i.e., \( \pi(T^*)\pi(T) \geq \pi(T)\pi(T^*) \)), and if \( E_1(T) \) is infinite, then \( R(\delta_T)^- + \{T^*\}' \) is not norm dense in \( \mathcal{L}(\mathcal{K}) \).

Proof. If \( T \) is an essentially hyponormal operator, it is known that \( R_e(T) = E_1(T) \) (see [6, p. 567]). Therefore, by Corollary 2, \( R(\delta_T)^- + \{T^*\}' \) is not norm dense in \( \mathcal{L}(\mathcal{K}) \).

Some open questions. If \( D = \begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix} \) on \( \mathcal{K} \oplus \mathcal{K} \), then \( \sigma(D) \) is obviously finite, and hence, by [1, Theorem 2.4], \( R(\delta_D)^- + \{D^*\}' = \mathcal{L}(\mathcal{K} \oplus \mathcal{K}) \). On the other hand, if
\[
D + K = \begin{pmatrix} I & 0 \\ 0 & K_{22} \end{pmatrix},
\]
where \( K_{22} \) is a diagonable compact operator with distinct entries on its diagonal, then \( R(\delta_{D+K})^- + \{D^* + K^*\}' \) is easily shown to be not dense in \( \mathcal{L}(\mathcal{K} \oplus \mathcal{K}) \) in the norm topology. Therefore, the following question is of some interest.
**Question 1.** For which nonnormal operator $T$, is it true that $R(Δ_T)^− + \{T^*\}'$ is norm dense in $L(ℋ)$?

For an arbitrary operator $A$ in $L(ℋ)$, it is in general not known whether $R(Δ_A) \cap \{A^*\}' = \{0\}$.

**Question 2.** If $T$ is a nonnormal operator, what are the necessary and sufficient conditions for $R(Δ_T) \cap \{T^*\}' = \{0\}$?

By a theorem of Kleinecke and Shirokov [4], [7], $R(Δ_A) \cap \{A\}'$ consists entirely of quasinilpotent operators for every operator $A$ in $L(ℋ)$. Therefore, if $T \in R(Δ_A) \cap \{A, A^*\}'$, then it is easy to show that $T^*T \in R(Δ_A) \cap \{A, A^*\}'$, and the quasinilpotence of $T^*T$ implies that $T = 0$. Therefore, $R(Δ_A) \cap \{A, A^*\}' = 0$ for every operator $A$ in $L(ℋ)$. Because every compact operator $K$ in $R(Δ_A)^− \cap \{A\}'$ is shown to be quasinilpotent by Theorem 2 in [3], $R(Δ_A)^− \cap \{A, A^*\}'$ does not contain any nonzero compact operator for every operator $A$ in $L(ℋ)$. However, there exists an operator $A$ such that $R(Δ_A)^− \cap \{A, A^*\}' \neq \{0\}$. In particular, if $1 \in R(Δ_A)^−$, then $R(Δ_A)^− \supseteq \{A\}'$: the existence of such an operator $A$ was established by J. Anderson in [2].

**Question 3.** If $T$ is a nonnormal operator, what are the necessary and sufficient conditions for $R(Δ_T)^− \cap \{T, T^*\}' \neq \{0\}$?

**BIBLIOGRAPHY**