HOMOMORPHIC IMAGES OF $\sigma$-COMPLETE BOOLEAN ALGEBRAS

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ABSTRACT. It is a well-known theorem of R. S. Pierce that, for every infinite cardinal $\alpha$, $\aleph_0^\alpha = \alpha$ if and only if there is a complete Boolean algebra $B$ s.t. $\text{card } B = \alpha$ (see [3, Theorem 25.4]). Recently, Comfort and Hager proved [1] that, for every infinite $\sigma$-complete Boolean algebra $B$, $(\text{card } B)^{\aleph_0} = \text{card } B$. We extend this result to the class of homomorphic images of $\sigma$-complete algebras, following closely Comfort's and Hager's proof. As a corollary, an improvement of Shelah's theorem on the cardinality of ultraproducts of finite sets [2] is derived (Theorem 2).

We denote the finite operations on a Boolean algebra $A$ by $+$, $\cdot$, and $-$, the infinite operations by $\Sigma$ and $\Pi$. If $a \in A$ and $a > 0$, $A\setminus a$ is the algebra $\{x \in A | x < a\}$. We write $\text{card } a$ for $\text{card } A\setminus a$ (if $a = 0$, $\text{card } a = 1$). A sequence $(a_n | n \in \omega)$ in $A$ is disjointed if $a_n \neq 0$ and $a_n \cdot a_m = 0$ for $n, m \in \omega, n \neq m$. For every sequence $(A_n | n \in \omega)$ of Boolean algebras, $\Pi_{n \in \omega} A_n$ is the product algebra.

Theorem 1. Let $B$ be a $\sigma$-complete Boolean algebra, $p$ an epimorphism from $B$ onto $A$ and $\alpha = \text{card } A \geq \aleph_0$. Then $\aleph_0^{\aleph_0} = \alpha$. 

Proof. We first prove three lemmas.

(a) Suppose $x, y \in B$ s.t. $x \leq y$, put $a = p(x)$ and $b = p(y)$. If $c \in A$ s.t. $a \leq c \leq b$, there is some $z \in B$ s.t. $x \leq z \leq y$ and $p(z) = c$. If $d \in A$ s.t. $a \cdot d = 0$, there is some $t \in B$ s.t. $x \cdot t = 0$ and $p(t) = d$.

Proof. Choose $z', t' \in B$ s.t. $p(z') = c$, $p(t') = d$; put $z = z' \cdot y + x$, $t = t' \cdot -x$.

(b) Suppose $I$ is an ideal of $A$ s.t. $\text{card } I = \text{card } A \geq \aleph_0$; every countable subset of $I$ has an upper bound in $I$ and for every $a \in I$, $\text{card } a^{\aleph_0}$ equals $2^{\aleph_0}$ or $\text{card } a$. Then $\aleph_0^{\aleph_0} = \alpha$. 

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1 This answers a question of M. Ziegler whether Theorem 1 allows to generalize Shelah's result to arbitrary reduced products of finite sets; the author's original proof only established Shelah's theorem.
Proof.

$$\alpha^{\aleph_0} = \text{card } \{ f : \aleph_0 \to I \} = \text{card } \{ f \text{ for some } a \in I, f : \aleph_0 \to A|a\}$$

$$\leq \text{card } I \cdot \max(2^{\aleph_0}, \sup_{a \in I} \text{card } a) \leq \alpha \cdot \max(2^{\aleph_0}, \alpha) = \alpha.$$ 

(c) Suppose $\langle a_n | n \in \omega \rangle$ is a disjointed sequence in $A$. Then $\text{card } \Pi_{n \in \omega} (A|a_n) \leq \alpha$.

Proof: We shall construct a one-one mapping $\phi$ from $P = \Pi_{n \in \omega} (A|a_n)$ to $A$. Let $f$ be a function (not necessarily a homomorphism) from $A$ to $B$ s.t. $p \circ f = \text{id}_A$. By (a) there is a disjointed sequence $\langle b_n | n \in \omega \rangle$ in $B$ s.t. $p(b_n) = a$ for $n \in \omega$. Define, for $\langle x_n | n \in \omega \rangle \in P$, 

$$\phi(\langle x_n | n \in \omega \rangle) = p \left( \sum_{n \in \omega} (f(x_n) \cdot b_n) \right).$$

Since $\phi(\langle x_n | n \in \omega \rangle) \cdot a_m = x_m$ for every $m \in \omega$, $\phi$ is one-one.

Returning to the proof of the theorem, suppose $\alpha < \alpha^{\aleph_0}$ and that $\alpha$ is the least cardinal providing a counterexample. By (c), $2^{\aleph_0} \leq \alpha$. Let $I = \{ a \in A | \text{card } a < \alpha \}$. $I$ is a proper ideal of $A$. If $a \in I$ and $0 < a$, $A|a$ is a homomorphic image of $A$, hence of $B$ and thus, by minimality of $\alpha$, $(\text{card } a)^{\aleph_0}$ equals $2^{\aleph_0}$ or card $a$.

The algebra $C = A/I$ is finite, for otherwise, by (a), there would exist a disjointed sequence $\langle a_n | n \in \omega \rangle$ in $A\setminus I$, and by (c), $\alpha^{\aleph_0} \leq \alpha$. Thus card $I = \alpha$. If every countable subset of $I$ has an upper bound in $I$, we have reached a contradiction. Thus, assume the contrary.

There is some $k \in \omega \setminus \{ 0 \}$ s.t. card $C = 2^k$ and some disjointed sequence $\langle a_j | j < k \rangle$ in $A\setminus I$. We may assume that $\Sigma_{j < k} a_j = 1$. There is some $j < k$ s.t. not every countable subset of $I \cap A|a_j$ has an upper bound in $I \cap A|a_j$. Put $A' = A|a_j$ and $I' = I \cap A'$. Then $A'$ is a homomorphic image of $A$ and hence of $B$, card $A' = \alpha$, $I'$ is a prime ideal in $A'$, card $I' = \alpha$, and for every $a \in I'$ s.t. $0 < a$, (card $A'|a$)$^{\aleph_0}$ equals $2^{\aleph_0}$ or card $A'|a$. We may and do assume that $A = A'$ and $I = I'$, i.e. that $I$ is a prime ideal of $A$. By the above assumption, there is a disjointed sequence $\langle c_n | n \in \omega \rangle$ in $I$ s.t. $\{ c_n | n \in \omega \}$ has no upper bound in $I$. Therefore, the ideal $K = \{ k \in A | k \cdot c_n = 0 \text{ for every } n \in \omega \}$ is a subset of $I$. By (c), card $\Pi_{n \in \omega} (A|c_n) \leq \alpha$. We actually have card $\Pi_{n \in \omega} (A|c_n) < \alpha$, for otherwise,

$$\alpha^{\aleph_0} = \left( \text{card } \Pi_{n \in \omega} (A|c_n) \right)^{\aleph_0} = \Pi_{n \in \omega} (\text{card } A|c_n)^{\aleph_0} = \Pi_{n \in \omega} (\text{card } A|c_n) = \alpha.$$
The range of the homomorphism \( f: A \rightarrow \prod_{n \in \omega} (A|c_n) \), defined by \( f(a) = (a \cdot c_n | n \in \omega) \) for \( a \in A \), thus has cardinality less than \( \alpha \), and its kernel \( K \) has cardinality \( \alpha \). We shall show that every countable subset of \( K \) has an upper bound in \( K \), which contradicts (b). Suppose \( \langle k_n | n \in \omega \rangle \) is a sequence in \( K \). Define, by (a), inductively \( \kappa_n \) and \( \delta_n \) in \( B \) s.t.

\[
p(\kappa_0) = k_0,
\]

\[
p(\delta_0) = -c_0, \quad \kappa_0 \leq \delta_0,
\]

\[
p(\kappa_{n+1}) = k_0 + \cdots + k_{n+1}, \quad \kappa_n \leq \kappa_{n+1} \leq \delta_n,
\]

\[
p(\delta_{n+1}) = -(c_0 + \cdots + c_{n+1}), \quad \kappa_{n+1} \leq \delta_{n+1} \leq \delta_n.
\]

Then \( d = p(\prod_{n \in \omega} \delta_n) \) is an upper bound of \( \{k_n | n \in \omega\} \) in \( K \).

**Corollary.** Suppose \( A \) is an infinite homomorphic image of some complete Boolean algebra. Then \( (\operatorname{card} A)^{\aleph_0} = \operatorname{card} A \).

**Remark.** Suppose \( A \) is a Boolean algebra satisfying

(1) if \( M, N \subseteq A \) are countable and \( m \cdot n = 0 \) for every \( m \in M, n \in N \),

then there exists \( a \in A \) s.t. \( m \leq a, a \cdot n = 0 \) for every \( m \in M, n \in N \).

Then \( (\operatorname{card} A)^{\aleph_0} = \operatorname{card} A \), provided \( \operatorname{card} A \geq \aleph_0 \). In fact, property (1) is all that is needed in the proof of Theorem 1. And it is clear that homomorphic images of \( \sigma \)-complete Boolean algebras and \( \aleph_1 \)-saturated Boolean algebras satisfy (1). This shows a close connection between Shelah’s method and ours.

We shall use two trivial facts about reduced products. Let \( I \) be a nonvoid set and \( F \) a filter on \( I \).

**Lemma 1.** Let \( X_i \) be an arbitrary nonvoid set for every \( i \in I \). Put \( J = \{i \in I | \operatorname{card} X_i = 1\}, K = I \setminus J \). If \( J \in F \), card \( \prod_{i \in J} X_i / F = 1 \). If \( J \notin F \), \( G = \{M \cap K | M \in F\} \) is a filter on \( K \) and there is a canonical isomorphism from \( \prod_{i \in K} X_i / F \) onto \( \prod_{i \in K} X_i / G \).

**Lemma 2.** Let \( X_i, Z_i \) be arbitrary sets for every \( i \in I \). Then there is a canonical isomorphism from \( \prod_{i \in I} (X_i \times Z_i) / F \) onto \( \prod_{i \in I} X_i / F \times \prod_{i \in I} Z_i / F \).

**Theorem 2.** Suppose \( I \neq \emptyset \), \( F \) is a filter on \( I \), \( n_i \in \omega \) for \( i \in I \), and \( \alpha = \operatorname{card} \prod_{i \in I} n_i / F \geq \aleph_0 \). Then \( \alpha^{\aleph_0} = \alpha \).

**Proof.** For every \( x = \langle x_i | i \in I \rangle \in \omega^I \), put \( c(x) = \operatorname{card} \prod_{i \in I} x_i / F \). We
first show:

(*) Suppose $x, y \in \omega^I$, $2 \leq x_i \leq y_i \leq 2x_i$ for every $i \in I$ and $c(x) \geq \aleph_0$. Then $c(y) = c(x)$.

We clearly may assume $y_i = 2x_i$ for every $i \in I$. But then we have, by Lemma 2, $c(y) = card 2^I/F \cdot c(x) = c(x)$.

To prove the theorem, let $n = (n_i | i \in I) \in \omega^I$ be given s.t. $c(n) \geq \aleph_0$. By Lemma 1, we may assume $2 \leq n_i$ for every $i \in I$. Define, for $i \in I$,

$$y_i = 2^{k_i + 1},$$

where $k_i \in \omega$ and $2^{k_i} \leq n_i < 2^{k_i + 1}$.

Since $n_i < 2^{k_i}$, we have, by (*), $\alpha = c(n) = c(y)$. Every $y_i$ may be considered as a finite (and hence, complete) Boolean algebra. Thus $\Pi_{i \in I} y_i$ carries, in a canonical way, the structure of a complete Boolean algebra and $A = \Pi_{i \in I} y_i / F$ the structure of a homomorphic image of a complete Boolean algebra. By the Corollary, $\alpha = c(y) = card A = \omega_0$.

Remarks. 1. It is easily seen that, if $F$ is an ultrafilter on $I$ and $y_i = 2^{k_i}$ for every $i \in I$ s.t. card $\Pi_{i \in I} y_i / F \geq \aleph_0$, then the algebra $A = \Pi_{i \in I} y_i / F$ includes a disjointed sequence $(a_n | n \in \omega)$ s.t. card $a_n = card A$ for every $n \in \omega$. This establishes, together with the lemma (c) in Theorem 1, a short proof of Shelah’s theorem.

2. The Corollary not only implies Theorem 2, but is, in fact, equivalent to it: let $B$ be complete, $p: B \rightarrow A$ an epimorphism and card $A \geq \aleph_0$. Since $B$ is an injective Boolean algebra, there is some set $I$ and some epimorphism $q$ from the power set of $I$ onto $B$. Put $F = \{x \subseteq I | p(q(x)) = 1\}$. Then $A$ is isomorphic to the reduced product $2^I/F$.

3. The question naturally arises whether there is also a simple way to prove Theorem 1 from the Corollary, i.e. whether every $\omega$-complete Boolean algebra is the homomorphic image of some complete Boolean algebra. This problem seems to be unsolved. The only fact I know is

Proposition (CH). Assume $A$ is a $\omega$-complete Boolean algebra s.t. card $A \leq 2^{\aleph_0}$. Then there is a complete Boolean algebra $B$ and a homomorphism $p$ from $B$ onto $A$.

Proof. If $A$ is finite, $A$ is complete. So, let $A$ be infinite. Thus, card $A = 2^{\aleph_0} = \aleph_1$. Let $A = \{a_\alpha | \alpha < \aleph_1\}$. Let $F$ be the free Boolean algebra on $\aleph_1$ free generators and let $B$ be $F^*$, the MacNeille completion of $F$. Since card $B = 2^{\aleph_0} = \aleph_1$, put $B = \{b_\alpha | \alpha < \aleph_1\}$. We construct a sequence $(p_\alpha | \alpha < \aleph_1)$ of homomorphisms from subalgebras of $B$ to $A$ s.t. $\alpha < \beta < \aleph_1$ implies $p_\alpha \subseteq p_\beta$. $\bigcup_{\alpha < \aleph_1} p_\alpha$ is a homomorphism from $B$ onto $A$ and each
$p_\alpha$ has countable domain. Let $p_0$ be the homomorphism from the two-element subalgebra of $B$ onto the two-element subalgebra of $A$. If $\lambda$ is a countable limit ordinal, put $p_\lambda = \bigcup_{\alpha < \lambda} p_\alpha$. Now suppose $\alpha < \aleph_1$ and $p_\alpha$ has been constructed s.t. $p_\alpha$ has countable domain.

Case 1. $\alpha$ is even. Let $b$ be the element of $B$ with the least subscript s.t. $b \notin \text{dom } p_\alpha$. Since dom $p_\alpha$ is countable and $A$ is $\sigma$-complete, an examination of the proof of Sikorski's extension theorem shows that $p_\alpha$ may be extended to the subalgebra of $B$ generated by $b$ and dom $p_\alpha$.

Case 2. $\alpha$ is odd. Let $a$ be the element of $A$ with least subscript s.t. $a \notin \text{rng } p_\alpha$. Since dom $p_\alpha$ is countable, there is some $b \in B$ s.t. $b$ is independent of dom $p_\alpha$. Hence $p_\alpha$ may be extended to the subalgebra of $B$ generated by $b$ and dom $p_\alpha$, putting $p_{\alpha+1}(b) = a$.

REFERENCES


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