SPECTRA OF COMPACT COMPOSITION OPERATORS
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ABSTRACT. Let \( \phi \) be holomorphic and map the open unit disk into itself, and let \( C_\phi : f \to f \circ \phi \) be the composition operator on \( H^2 \) generated by \( \phi \). If \( C_\phi \) is a compact operator then (1) \( \phi(z_0) = z_0 \) for some \( z_0 \in D \); (2) \( \sigma(C_\phi) = \{ \phi^n(z_0) \}^\infty : n = 0, 1, 2, \ldots \cup \{0\} \).

Let \( \phi \) be holomorphic and nonconstant in the open unit disk \( D \) and suppose \( |\phi(z)| < 1 \) for \( z \in D \). For \( f \) holomorphic in \( D \), define the composition operator \( C_\phi \) by \( (C_\phi f)(z) = (f \circ \phi)(z) = f(\phi(z)) \). Then \( C_\phi \) is a bounded linear operator on the Hilbert space \( H^2 \) [6].

In this paper, we describe the spectrum of \( C_\phi \) in the case in which \( C_\phi \) is compact. (For a discussion of geometric conditions which guarantee compactness, see [7].) To describe the spectrum we prove a theorem which may be of independent interest: that if \( C_\phi \) is compact, \( \phi \) has a fixed point in \( D \).

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The reproducing kernel for \( H^2 \), \( k_\zeta(z) = (1 - \overline{\zeta}z)^{-1}, \zeta \in D \), is characterized by the property that for \( f \in H^2 \), \( (f, k_\zeta) = f(\zeta) \). The function \( k_\zeta \) is itself an \( H^2 \) function, of norm \( (1 - |\zeta|^2)^{-1/2} \). In terms of these functions, we can characterize the composition operators among operators on \( H^2 \).

Theorem 1. Let \( A \) be an operator on \( H^2 \). Then \( A \) is a composition operator if and only if the image of every kernel function under \( A^* \) is a kernel function. If \( \zeta \in D \), \( C_\phi^* k_\zeta = k_\phi(\zeta) \).

Proof. If \( A = C_\phi \) is a composition operator, then

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\[(f, A^*k_\zeta) = (C_\phi f, k_\zeta) = f(\phi(\zeta)) = (f, k_\phi(\zeta)),\]

so \[A^*k_\zeta = k_\phi(\zeta)\] for every \(\zeta \in \mathbb{D} \).

Conversely, if \(A^*\) takes each kernel function \(k_\zeta\) into a kernel function \(k_{\phi(\zeta)}\), then if \(f \in H^2\),

\[(Af)(\zeta) = (Af, k_\zeta) = (f, A^*k_\zeta) = (f, k_{\phi(z)}) = f(\phi(\zeta)).\]

Letting \(f\) be the identity function \(f(z) = z\), we see \(\phi\) is holomorphic and \(A = C_\phi\).

**Theorem 2.** If \(C_\phi\) is a compact composition operator, then \(\phi(z)\) has a fixed point in \(\mathbb{D}\).

**Proof.** Since \(C_\phi\) is compact, \(\phi\) is not a rotation about a fixed point \([7]\). By the Wolff-Denjoy theorem \([2], [8]\), there is a point \(\alpha \in \overline{\mathbb{D}}\) so that for every \(z \in \mathbb{D}\), the iterates \(\phi_n(z)\) converge to \(\alpha\), where \(\phi_0(z) = z, \phi_{n+1}(z) = \phi(\phi_n(z))\). If \(|\alpha| < 1\), then \(\phi(\alpha) = \alpha\).

Suppose, by way of contradiction, that \(|\alpha| = 1\). Since \(\lim_{n \to \infty} \phi_n(0) = \alpha\), for infinitely many \(n\), \(|\phi_{n+1}(0)| > |\phi_n(0)|\). Choose points \(\zeta_n = \phi_n(0)\) from among these values of \(n\). Let \(k_{\gamma_n}\) be the kernel function for \(\zeta_n\). Then \(C_\phi^*k_{\gamma_n} = k_{\gamma_{n+1}}\). Because \(C_\phi^*\) is compact, the images of \(k_{\gamma_n}/\|k_{\gamma_n}\|\) under \(C_\phi^*\) have a subsequence \(\{g_j\}\) which converges in norm. Since

\[\|g_j\| = \|k_{\gamma_{n+1}}/\|k_{\gamma_n}\| = (1 - |\phi_n(0)|^2)^{1/2}/(1 - |\phi_{n+1}(0)|^2)^{1/2} > 1,\]

the norm of the limit function \(g\) is at least 1.

On the other hand,

\[(g, g_j) = (1 - |\zeta_n|^2)^{1/2}(g, k_{\gamma_{n+1}}) = \left(\frac{1 - |\zeta_n|^2}{1 - |\zeta_{n+1}|^2}\right)^{1/2} (1 - |\zeta_{n+1}|^2)^{1/2}g(\zeta_{n+1})\]

\[< \|C_\phi\|o(1)\]

since \(\lim_{|z| \to 1} (1 - |z|)^{1/2}f(z) = 0\) for \(f \in H^2\) \([3, \text{Theorem 5.9}]\). So \(\|g\|^2 = \lim (g, g_j) = 0\). This contradiction shows \(|\alpha| < 1\).

**Corollary.** If \(C_\phi^N\) is compact for some \(N > 0\), then \(\phi(z)\) has a fixed point in \(\mathbb{D}\).
Proof. Because $C^N_{\phi} = C^N_{\phi'}$, for some $a \in D$, $\phi'_N(a) = a$, and thus $\phi_{kN}'(a) = a$ for positive integers $k$. A periodic sequence converges only if constant, so $\phi(a) = a$.

Theorem 3. If $C^N_{\phi}$ is compact for some $N > 0$, then $\sigma(C_{\phi'}) = \{\phi'(z_0)^n : n = 1, 2, \ldots \} \cup \{0, 1\}$, where $z_0$ is the fixed point of $\phi$.

Proof. By an operator similarity, we may assume the fixed point is at 0. For if $r$ is a linear fractional transformation of $D$ onto $D$ taking $z_0$ to 0, then

$$C_rC_{\phi}C_r^{-1} = C_{r\circ \phi \circ r^{-1}},$$

and $r \circ \phi \circ r^{-1}(0) = 0$. By the chain rule, $(r \circ \phi \circ r^{-1})'(0) = \phi'(z_0)$.

The matrix of $C^*_{\phi}$ with respect to the basis $\{z^k\}$ is upper triangular with diagonal entries $1$, $\overline{\phi'(0)}$, $\overline{\phi'(0)^2}$, $\overline{\phi'(0)^3}$, $\ldots$ $[1]$, so $\phi'(0)^n \in \sigma(C_{\phi})$. Thus $\{\phi'(0)^n : n = 1, 2, \ldots \} \cup \{0, 1\} \subseteq \sigma(C_{\phi})$.

It remains to show that the only possible eigenvalues for $C_{\phi}$ are 1 and $\phi'(0)^n$ for some $n$. This is essentially due to Koenig [4]. If $f \circ \phi = \lambda f$, then $f(0) = \lambda f(0)$, so if $\lambda \neq 1$, $f(0) = 0$. Write $f(z) = z^m g(z)$, where $g$ is holomorphic in $D$ and $g(0) \neq 0$. It follows that $\phi(z)^m g(\phi(z)) = \lambda z^m g(z)$, and thus $(\phi(z)/z)^m g(\phi(z)) = \lambda g(z)$. The value of $\phi(z)/z$ at $z = 0$ is $\phi'(0)$, so $\lambda = \phi'(0)^m$.

Since the nonzero spectrum of a power-compact operator consists of eigenvalues, the theorem is proved.

REFERENCES


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