THE Baire ORDER OF THE FUNCTIONS
CONTINUOUS ALMOST EVERYWHERE. II

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ABSTRACT. Let $S$ be a complete and separable metric space and $\mu$ a $\sigma$-finite, complete Borel measure on $S$ with $\mu(S) > 0$. Let $\mathcal{F}$ be the family of all real-valued functions defined on $S$ whose set of points of discontinuity is of $\mu$-measure 0. Let $B_{\alpha}(\mathcal{F})$ be the functions of Baire’s class $\alpha$ generated by $\mathcal{F}$. It is shown that $B_{1}(\mathcal{F}) = B_{2}(\mathcal{F})$ if and only if $\mu$ is a purely atomic measure whose set of atoms forms a scattered subset of $S$ and that if $B_{1}(\mathcal{F}) \neq B_{2}(\mathcal{F})$, then the Baire order of $\mathcal{F}$ is $\omega_1$; in other words, if $0 < \alpha < \omega_1$, then $B_{\alpha}(\mathcal{F}) \neq B_{\alpha+1}(\mathcal{F})$. This answers a generalized version of a problem raised by Sierpinski and Felsztyn. An example is given of a normal space with Borel order 2 and Baire order $\omega_1$.

Sierpinski and Felsztyn in the first volume of Fundamenta Mathematicae raised the following problem:

\((*)\) Is there a function of Baire’s class 2 on the unit interval which is not the pointwise limit of a sequence of functions each continuous almost everywhere [5]?

There is a discussion of this problem in the appendix of the 1937 edition of the first volume. This problem was solved by Zalcwasser and Kantorovitch. Also, see [4].

In Theorem 4 of [4], the author shows that for each countable ordinal $\alpha$, there is a function of Baire’s class $\alpha + 1$ which is not in the $\alpha$ class generated by the functions continuous almost everywhere. Therefore, the answer to \((*)\) and to a generalized version of \((*)\) is yes.

This paper contains a number of generalizations of results contained in [4].

Definitions and notation. If $X$ is a topological space and $\mu$ is a complete Borel measure on $X$, $A$ is a subset of $X$, and $B$ is a subset of $A$, then

(a) $\Phi(A, \mu)$ will denote the family of all real-valued functions defined on $A$ whose set of points of discontinuity is of $\mu$-measure zero, and

(b) $\Phi(A, B)$ will denote the family of all real-valued functions defined
on $A$ which are continuous at each point of $B$.

If $X$ is a set and $\Phi$ is a family of real-valued functions defined on $X$, then $B_0(\Phi)$ will denote $\Phi$ and for each ordinal $\alpha$, $\alpha > 0$, $B_\alpha(\Phi)$ will denote the family of all pointwise limits of sequences from $\bigcup_{\gamma < \alpha} B_\gamma(\Phi)$. Of course, $B_{\omega_1}(\Phi) = \bigcup_{\alpha < \omega_1} B_\alpha(\Phi)$ and thus, $B_{\omega_1}(\Phi) = B_{\omega_1 + 1}(\Phi)$. The first ordinal $\alpha$ for which $B_\alpha(\Phi) = B_{\alpha + 1}(\Phi)$ will be called the Baire order of $\Phi$.

The unit interval will be denoted by $I$.

Recall that a subset $M$ of a topological space is said to be scattered if there is no subset of $M$ which is dense in itself. Also, in this paper the Borel sets form the $\sigma$-algebra generated by the open sets and a measure $\mu$ is regular means $\mu(E) = \sup \{\mu(F) : F \subset E\} = \inf \{\mu(U) : U$ is open and $E \subset U\}$, for each $\mu$-measurable set $E$.

**Theorem 1.** Suppose $\mu$ is a finite, positive complete Borel measure on $I$ and $\mu(I) > 0$. If $\mu$ is not a purely atomic measure whose set of atoms forms a scattered set, then the Baire order of $(I, \mu)$ is $\omega_1$.

**Proof.** Let $M$ be the set of all atoms of the measure $\mu$. Either (1) the countable set $M$ contains a dense in itself subset $K$, or (2) $\mu(I - M) > 0$. If the first case holds, then $K$ is a perfect subset of $I$ such that if an open set $U$ meets $K$, then $\mu(K \cap U) > 0$. If the second case holds, then there is a perfect set lying in $I - M$ such that if an open set meets $P$, then $\mu(P \cap U) > 0$.

It is easy to check that one may now proceed exactly as in [4], and conclude that the Baire order of $\Phi(I, \mu)$ is $\omega_1$.

**Theorem 2.** Let $K$ be a subset of a metric space $S$ and let $D$ and $A$ be $G_\delta$ subsets of $S$ containing $K$ with $K \subset D \subset A$. Then

(a) if $\alpha > 0$, each function in $B_\alpha^*(\Phi(D, K))$ has an extension to a function in $B_\alpha(\Phi(A, K))$,

(b) the Baire order of $\Phi(D, K)$ is no more than the Baire order of $\Phi(A, K)$,

(c) if the Baire order of $\Phi(D, K)$ is $> 0$, then $\Phi(A, K)$ and $\Phi(D, K)$ have the same order.

**Proof.** (a) If $f \in B_\alpha^*(\Phi(D, K))$ and $\alpha > 0$, then by Theorem 3 of [2], there is a function $g$ of Baire's class $\alpha$ (in other words, $g \in B_\alpha(\Phi(D, D))$ such that $M = \{x | f(x) \neq g(x)\}$, is a subset of an $F_\sigma$ set, $W$, with respect to $D$ and $W$ does not intersect $K$.

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\[ \hat{f}(x) = \begin{cases} f(x), & x \in D, \\ \hat{g}(x), & x \in A - D. \end{cases} \]

The set of all \( x \) such that \( \hat{f}(x) \neq \hat{g}(x) \) is \( M \). Let \( W = \bigcup_{n=1}^{\infty} F_n \), where for each \( n \), \( F_n \) is closed with respect to \( D \) and let \( \hat{F}_n \) be the closure of \( F_n \) in \( A \). Then \( M \subseteq \hat{W} = \bigcup_{n=1}^{\infty} \hat{F}_n \) and \( \hat{W} \) is an \( F_\sigma \) set with respect to \( A \) which does not meet \( K \). Thus, by Theorem 3 of [2], \( \hat{f} \in B_\alpha(\Phi(A, K)) \).

(b) It may be shown by transfinite induction, that for all \( \alpha, 0 \leq \alpha \), if \( f \in B_\alpha(\Phi(A, K)) \), then the restriction of \( f \) to \( D \) is in the family \( B_\alpha(\Phi(D, K)) \). From this we see that if \( f \) is exactly of class \( B_\gamma + 1(\Phi(D, K)) \) (\( f \in B_\alpha(\Phi(D, K)) - \bigcup_{\gamma < \alpha} B_\gamma(\Phi(D, K)) \)), then no extension of \( f \) to \( A \) can be of lower class with respect to \( \Phi(A, K) \). Thus, the Baire order of \( \Phi(D, K) \) is no more than the Baire order of \( \Phi(A, K) \).

(c) Suppose the Baire order of \( \Phi(A, K) \) is greater than \( \gamma \), the Baire order of \( \Phi(D, K) \). Let \( f \) be a function of exactly class \( B_{\gamma + 1}(\Phi(A, K)) \) and let \( h \) be the restriction of \( f \) to \( D \). Then \( h \in B_{\gamma + 1}(\Phi(D, K)) \) and therefore \( h \in B_{\gamma}(\Phi(D, K)) \). Since \( \gamma > 0 \), by part (a), there is an extension \( \hat{h} \) of \( h \) to \( A \) which is in \( B_{\gamma}(\Phi(A, K)) \). Let \( M = \{ x | \hat{h}(x) \neq f(x) \} \). The set \( M \) is a subset of \( A - D \). But, \( A - D \) is an \( F_\sigma \) set with respect to \( A \) which does not meet \( K \). It follows from Theorem 3 of [2], that \( f \in B_{\gamma}(\Phi(A, K)) \). This contradiction completes the argument for part (c).

Theorem 3. Let \( A \) and \( D \) be \( G_\delta \) subsets of a metric space \( S \) with \( D \subseteq A \). Let \( \mu \) be a finite regular complete Borel measure defined on \( A \). If \( \mu(A - D) = 0 \), then

(a) if \( \alpha > 0 \), each function in \( B_\alpha(\Phi(D, \mu)) \) has an extension to a function in \( B_\alpha(\Phi(A, \mu)) \),

(b) the Baire order of \( \Phi(D, \mu) \) is no more than the Baire order of \( \Phi(A, \mu) \), and

(c) if the Baire order of \( \Phi(D, \mu) \) is \( > 0 \), then \( \Phi(A, \mu) \) and \( \Phi(D, \mu) \) have the same order.

The proof of this theorem follows the corresponding proofs of Theorem 2.

Theorem 4. Let \( R \) be the set of all rational numbers in \( I \), let \( B \) be a \( G_\delta \) subset of \( I \) containing \( R \). Then the Baire order of \( \Phi(B, R) \) is \( \omega_1 \).

Proof. Let \( \mu \) be a finite, complete Borel measure on \( I \) such that \( \mu \) is purely atomic and \( R \) is the set of all atoms of \( \mu \). Then, the family \( \Phi(I, R) \) is \( \Phi(I, \mu) \). It is easy to see that the Baire order of \( \Phi(B, R) \) is not \( 0 \). There-
fore, by Theorem 2 (c), the Baire order of $\Phi(B, R)$ is $\omega_1$.

**Theorem 5.** Let $K$ be a countable dense in itself subset of a complete and separable metric space $S$ and let $A$ be a $G_\delta$ subset of $S$ containing $K$. Then the Baire order of $\Phi(A, K)$ is $\omega_1$.

**Proof.** Let $\phi$ be a homeomorphism of $K$ with the set of all rational numbers in the unit interval $I$ [1, p. 287]. Let $\hat{\phi}$ be an extension of $\phi$ defined on a $G_\delta$ set $B$ containing $K$ to a $G_\delta$ set, $\hat{\phi}(B)$, in $I$ such that $\hat{\phi}$ is a homeomorphism of $B$ and $\hat{\phi}(B)$ [1, p. 429].

It follows easily by transfinite induction that $f \in B_\alpha(\Phi(A \cap B, K))$ if and only if $f \circ \hat{\phi}^{-1} \in B_\alpha(\hat{\phi}(A \cap B), R))$. Therefore, the order of the family $\Phi(A \cap B, K)$ is $\omega_1$ by Theorem 3. Thus, the Baire order of the family $\Phi(A, K)$ is $\omega_1$ by Theorem 2 (c).

**Theorem 6.** Let $M$ be a subset of a complete and separable metric space. If $M$ contains a perfect set, then the Baire order of $\Phi(S, M)$ is $\omega_1$. If $M$ is countable, then (1) the Baire order of $\Phi(S, M)$ is $\leq 1$, if $M$ is scattered and (2) the Baire order of $\Phi(S, M)$ is $\omega_1$, if $M$ is not scattered.

**Proof.** Suppose $M$ contains a perfect set $K$. Since $\Phi(K, K)$ is the space of all real valued continuous functions defined on $K$, it follows that the Baire order of $\Phi(K, K)$ is $\omega_1$. Also, for each $\alpha, 0 \leq \alpha$, each function in $B_\alpha(\Phi(K, K))$ has an extension to a function in $B_\alpha(\Phi(S, S))$ [1, p. 434] and thus to a function in $B_\alpha(\Phi(S, M))$. It follows that if $f \in B_\alpha(\Phi(K, K))$ but to none of the preceding classes, then any extension of $f$ to a function in $B_\alpha(\Phi(S, M))$ cannot belong to any class $B_\gamma(\Phi(S, M))$, $\gamma < \alpha$.

Therefore, the order of $\Phi(S, M)$ is $\omega_1$.

Now, suppose $M$ is countable.

Case 1. The set $M$ is scattered. In this case, Theorem 2 of [3] states that the Baire order of $\Phi(S, M)$ is $\leq 1$.

Case 2. The set $M$ is not scattered. Let $K$ be the dense in itself kernel of $M$.

If $M$ is $K$, then by Theorem 5 the Baire order of $\Phi(S, M) = \Phi(S, K)$ is $\omega_1$.

If $K$ is a proper subset of $M$, then the set $M - K$ is scattered. Therefore $M - K$ is an $F_\sigma$ set [1, p. 258]. Then $S - (M - K)$ is a $G_\delta$ set containing $K$ and the Baire order of $\Phi(S - (M - K), K)$ is $\omega_1$ by Theorem 5.

If $f$ is of exactly class $B_{\alpha+1}(\Phi(S - (M - k), K))$, $\alpha > 0$, then there is a function $g$ of Baire's class $\alpha + 1$ on $S - (M - K)$ such that the set $M = \{\inf(x) \neq g(x)\}$ is a subset of a set $W$ which is an $F_\sigma$ set with respect to...
S - (M - K). Let \( \hat{g} \) be an extension of \( g \) to \( S \) of Baire's class \( \alpha + 1 \). Then obviously, \( g \in B_{\alpha+1}(\Phi(S, M)) \). Assume \( g \in B_\alpha(\Phi(S, M)) \). Then there is a function \( h \) in Baire's class \( \alpha \) on \( S \) such that the set \( M_1 = \{ x : \hat{g}(x) \neq h(x) \} \) is a subset of an \( F_\sigma \) set \( W_1 \) in \( S \) such that \( W_1 \) does not intersect \( K \) [2, Theorem 3]. But, then \( l \), the restriction of \( h \) to \( S - (M - K) \), is a function of Baire's \( \alpha \) on \( S - (M - K) \) and the set of all \( x \) such that \( l(x) \neq f(x) \) is a subset of \( W_1 \cap (S - (M - K)) \), which is an \( F_\sigma \) set in \( S - (M - K) \) which does not meet \( K \). Therefore, by Theorem 3 of [2], \( f \) is in \( B_\alpha(\Phi(S - (M - K), K)) \). This contradiction proves that the order of \( \Phi(S, M) \) is \( \omega_1 \).

**Questions.** Is there a subset \( M \) of \( I \) such that the Baire order of \( \Phi(I, M) \) is \( 2 \)? For each ordinal \( \alpha \), \( 2 \leq \alpha < \omega_1 \), is there a subset \( M \) of \( I \) such that the Baire order of \( \Phi(I, M) \) is \( \alpha \)?

**Theorem 6.** Let \( \mu \) be a finite regular Borel measure defined on the space \( N \) consisting of all irrational numbers between 0 and 1. If \( \mu \) has no atoms and \( \mu(N) > 0 \), then the order of \( \Phi(I, \mu) \) is \( \omega_1 \).

**Proof.** Let \( \hat{\mu} \) be the unique extension of \( \mu \) to a complete Borel measure defined on \( I \) such that \( \hat{\mu}(I - N) = 0 \). Then \( \hat{\mu}(I) > 0 \) and \( \hat{\mu} \) has no atoms. Therefore the Baire order of \( \Phi(I, \mu) \) is \( \omega_1 \). Therefore, by Theorem 2, the Baire order of \( \Phi(N, \mu) \) is \( \omega_1 \).

**Theorem 7.** Let \( \mu \) be a \( \sigma \)-finite regular Borel measure defined on a complete and separable metric space \( S \) with \( \mu(S) > 0 \). Then (1) the order of \( \Phi(S, \mu) \) is \( \leq 1 \) if and only if \( \mu \) is purely atomic and the set of atoms of \( \mu \) forms a scattered set, and (2) the order of \( \Phi(S, \mu) \) is \( \omega_1 \), if \( \mu \) does not meet the conditions described in 1.

**Proof.** Part (1) of the conclusion is Theorem 3 of [3].

Let \( \{ K_n \}_{n=1}^\infty \) be a sequence of disjoint Borel sets of finite \( \mu \)-measure filling up \( S \). Let \( \mu_n(A) = \mu(A \cap K_n) \), for each \( n \) and each \( \mu \)-measurable set \( A \). Let

\[
\nu = \sum_{n=1}^{\infty} \frac{2^{-n}}{\mu_n(K_n)} + 1 \mu_n.
\]

Then \( \nu \) is a finite regular Borel measure on \( S \) and a subset \( E \) of \( S \) is of \( \mu \)-measure 0 if and only if \( \nu(E) = 0 \).

Let \( \nu = \nu_d + \nu_s \), where \( \nu_d \) is purely atomic and \( \nu_s \) has no atoms. Let \( M \) be the set of atoms of \( \nu_d \). Of course, \( M \) is the set of atoms of \( \mu \). It follows from part (1) of the conclusion that either \( M \) is not scattered or \( \nu(S - M) > 0 \).
Case 1. Suppose $M$ is not scattered. Let $K$ be the dense in itself kernel of $M$ and let $A$ be a $G_{\delta}$ set containing $K$ such that $\nu(A) = \nu(K)$. Then $\nu(A - K) = 0$ and $\Phi(A, \nu) = \Phi(A, K)$. Therefore, by Theorem 5, the order of $\Phi(A, \nu)$ is $\omega_1$ and by Theorem 3 the order of $\Phi(S, \nu) = \Phi(S, \mu)$ is $\omega_1$.

Case 2. Suppose $\nu(S - M) > 0$.

Let $J$ be a perfect set lying in $S - M$ such that if an open set $U$ meets $J$, then $\nu(J \cap U) > 0$. Let $\{y_n\}_{n=1}^{\infty}$ be a dense subset of $J$ and for each $n$, let $\{\delta_n\}_{p=1}^{\infty}$ be a decreasing sequence of positive numbers converging to zero such that $\nu(B(y_n, \delta_n) - B(y_n, \delta_n)) = 0$, where $B(y_n, \delta_n)$ is the ball with center $y_n$ and radius $\delta_n$. Let $Q$ be the union of all the sets $B(y_n, \delta_n) - B(y_n, \delta_n)$. It follows that $Q \cap J$ is an $F_\sigma$ subset of $J$ with $\nu(Q) = 0$ such that $J - Q$ is 0-dimensional.

Let $W = J - Q$. Then $W$ is a dense in itself 0-dimensional $G_{\delta}$ set lying in $J$. By Theorem 3, the Baire order of $\Phi(J, \nu)$ is the same as the order of $\Phi(W, \nu)$.

Let $\phi$ be a homeomorphism of $W$ onto $N$, the set of all irrational numbers between 0 and 1 [1, p. 441], and for each $\nu$-measurable set $E$ lying in $W$, let $\lambda(\phi(E)) = \nu(E)$. It follows that $\lambda$ is a complete Borel measure on $N$ and a function $f$ is in the class $B_\alpha(\Phi(N, \lambda))$ if and only if $f \circ \phi$ is in the class $B_\alpha(\Phi(W, \lambda))$. By Theorem 5, the Baire order of $\Phi(N, \lambda)$ is $\omega_1$. Thus, the order of $\Phi(J, \nu)$ is $\omega_1$.

Finally, if $h \in B_\alpha(\Phi(S, \nu))$, then the restriction of $h$ to $J$ is in $B_\alpha(\Phi(J, \nu))$. Also, if $\alpha > 0$ and $f \in B_\alpha(\Phi(J, \nu))$, then there is a function $g$ of Baire’s class $\alpha$ defined on $J$ such that the set $M$ of all $x$ such that $g(x) \neq f(x)$ is a subset of an $F_\sigma$ set $T$ with respect to $J$.

Let $\hat{g}$ be an extension of Baire’s class $\alpha$ to all of $S$ [1, p. 434], let $\hat{f}(x) = f(x), x \in J$, and $\hat{f}(x) = g(x), x \in S - J$. Then the set of all $x$ such that $\hat{f}(x) \neq \hat{g}(x)$ is a subset of $T$. Since $T$ is an $F_\sigma$ set with respect to $J$, $T$ is an $F_\sigma$ set in $S$ of $\nu$-measure zero. Therefore, by Theorem 3 of [3], $f \in B_\alpha(\Phi(S, \nu))$.

From the above considerations, it follows that the order of $\Phi(S, \nu)$, which is $\Phi(S, \mu)$, is $\omega_1$.

Theorem 8. There is a hereditarily paracompact space which has Borel order 2 and Baire order $\omega_1$.

Proof. Let $X$ be the unit interval and let a subset $W$ of $X$ be open if and only if $W = U \cap V$, where $U$, is open and $V$, is any subset of $X - R$,
where $R$ is the rationals. The space $X$ is hereditarily paracompact [6].

S. Willard in [7] shows that every Borel subset of $X$ is a $G_{δσ}$ set in $X$. If $f ∈ C(X)$, then $f$ is continuous in the usual topology at each point of $R$. Thus, by Theorem 4, $X$ has Baire sets of arbitrarily high class.

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