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A CHARACTERIZATION OF THE KERNEL OF A CLOSED SET

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ABSTRACT. Let S be a closed subset of some linear topological space such that int ker $S \neq \emptyset$ and ker $S \neq S$. Let $\mathcal{C}$ denote the collection of all maximal convex subsets of S and, for any fixed $k \geq 1$, let $\mathfrak{M} = \{A_1 \cup \cdots \cup A_k : A_1, \ldots, A_k$ distinct members of $\mathcal{C}\}$. Then $\mathfrak{M} \neq \emptyset$ and $\bigcap \mathfrak{M} = \ker S$.

If $\mathcal{C}$ is the collection of all maximal convex subsets of some set S, it is easy to show that $\bigcap \mathcal{C} = \ker S$. This paper provides an interesting and perhaps surprising analogue of this well-known result. Throughout the paper, conv S, int S, and ker S will be used to denote the convex hull, interior, and kernel, respectively, for the set S.

Further, we will make use of these familiar definitions: For points x, y in a set S, we say x *sees* y *via* S if and only if the corresponding segment $[x, y]$ lies in S. A subset T of S is said to be a *visually independent subset of* S if and only if for every x, y in T, $x \neq y$, x does not see y via S.

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Theorem 1. *Let S be a closed subset of some linear topological space such that* int ker $S \neq \emptyset$ *and* ker $S \neq S$. *Let $\mathcal{C}$ denote the collection of all maximal convex subsets of S and, for any fixed $k \geq 1$, let $\mathfrak{M} = \{A_1 \cup \cdots \cup A_k : A_1, \ldots, A_k$ distinct members of $\mathcal{C}\}$. Then $\mathfrak{M} \neq \emptyset$ and $\bigcap \mathfrak{M} =$* ker S.

Proof. It is clear that ker $S \subseteq \bigcap \mathfrak{M}$, since ker S lies in every member of $\mathcal{C}$. To prove the reverse inclusion, we show that if $x \in S$ and $x \notin$ ker S, there are infinitely many distinct members of $\mathcal{C}$ which fail to contain x. Since $x \notin$ ker S, we may select p in S with $[p, x] \not\subseteq S$. Also, select z in int ker $S \neq \emptyset$. Clearly z, p, x are not collinear. Because S is closed, $[p, z] \subseteq S$ and $[p, x] \not\subseteq S$, there is some point w on $[z, x)$ such that p sees w via S and p sees no point of $(w, x]$ via S. Also, since $z \in$ int ker S, w lies in the open interval (z, x), and conv$\{p, z, w\} \subseteq S$. Similarly, there is a point y on (z, p) such that x sees y via S, x sees no point of $(y, p]$ via S, and conv$\{x, z, y\} \subseteq S$. Let q denote the point of intersection of (p, w) with (x, y). There are two cases to consider.

Case 1. Assume for the moment that no point of $[p, q)$ sees any point of $[x, q)$ via S. Consider the family of segments $[a, b]$ supporting conv$\{p, q, x\}$ at q, with a on $[p, y)$ and b on $[w, x)$. Each of these segments lies in a maximal convex subset of S not containing x, and no two segments lie in the same maximal convex subset. Hence there are infinitely many maximal convex subsets of S not containing x, and $x \notin \bigcap \mathfrak{M}$, the desired result.

Case 2. If some point of $[p, q)$ sees some point of $[x, q)$ via S, select points p_2 and x_2 having this property, with p_2 on $[p, q)$ and x_2 on $[x, q)$. Clearly $p_2 \neq p$ and $x_2 \neq x$, and we may select p_2, x_2 so that no point of $[p, p_2)$ sees any point of $[x, x_2)$ via S. Repeat an earlier argument to find points w_2 on $[x_2, q)$, y_2 on $[p_2, q)$ such that p_2 sees w_2 via S and p_2 sees no point of $(w_2, x]$ via S, x_2 sees y_2 via S and x_2 sees no point of $(y_2, p]$ via S.

Without loss of generality, we assume that $p_2 \neq y_2$ (for otherwise the following argument may be suitably adapted using p, p_2, x_2 in place of p_2, q_2, x_2, respectively). Let q_2 denote the point of intersection of $[p_2, w_2]$ with $[x_2, y_2]$. It is clear that x sees no point on $[p_2, q_2] \cup (x_2, q_2]$. In case no point of $[p_2, q_2)$ sees any point of $[x_2, q_2)$ via S, we may repeat the argument of Case 1 to obtain an infinite collection of segments supporting conv$\{p_2, q_2, x_2\}$ at q_2, each of which lies in a maximal convex subset of S not containing x, and no two of which lie in the same maximal convex subset of S, finishing the proof.

Otherwise, some point of $[p_2, q_2)$ sees some point of $[x_2, q_2)$ via S, and we repeat the previous argument to obtain points p_3, x_3, q_3. Furthermore, x cannot see x_3 via S. Continuing inductively, if for some n, no point of $[p_n, q_n)$ sees any point of $[x_n, q_n)$ via S, then the argument of Case 1 yields the desired result. If no such n exists, then the infinite set of points $\{x_{2n+1}: n \geq 1\}$ is a visually independent subset of S, no point of which sees x via S. To each point x_{2n+1} we may associate a distinct maximal convex subset of S not containing x. Therefore, $x \notin \bigcap \mathfrak{M}$. This completes Case 2 and the proof of the Theorem.

To see that the full hypothesis of Theorem 1 is required, consider the following example.

Example. For $k \geq 2$, let $x_1, \ldots, x_k$ denote k distinct points of some line L, with $x_1 < x_2 < \cdots < x_k$, and let y be a point not on L. If $S = \operatorname{int}(\operatorname{conv}\{x_1, x_k, y\}) \cup \{x_1, \ldots, x_k\}$, then S is not closed, S has exactly k maximal convex subsets, and the corresponding set $\bigcap \mathfrak{M}$ is all of S.

Similarly, if S is any collection of $k \geq 2$ distinct lines intersecting in a common point, then $\operatorname{int}(\ker S) = \varnothing$, S has exactly k maximal convex subsets, and $\bigcap \mathfrak{M} = S$.

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