ALGEBRAS SATISFYING CONGRUENCE RELATIONS
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ABSTRACT. It is shown that the classical nonassociative algebras which have an identity element can be defined in terms of congruence relations modulo the base field.

1. Introduction. In an earlier note [2] two of the authors have shown that if an algebra \( A \) with identity element \( 1 \) over a field \( F \) satisfies the property that \( (xy)z - x(yz) \in F \cdot 1 \) for all \( x, y, z \) in \( A \), then \( A \) is an associative algebra. Here we consider the similar question for the classical nonassociative algebras. Recall that an alternative algebra is one which satisfies \( x(yz) - (xy)z = 0 \) for all elements \( x, y \) and \( \alpha \) (linear) Jordan algebra is a commutative algebra over a field of characteristic \( \neq 2 \) which satisfies \( (xy)x - x(yx) = 0 \) for all elements \( x, y \). In our main results we show that if \( A \) is an algebra with identity element \( 1 \) over a field \( F \) such that \( xy - yx \in F \cdot 1 \) and \( ixy)x - x(yx) \in F \cdot 1 \) for all elements \( x \) and \( y \), then \( A \) is a Jordan algebra. Also if the characteristic of \( F \) is not \( 3 \) and if \( x(xy) - x^2y \in F \cdot 1 \) and \( ixy)x - yx \in F \cdot 1 \) for all elements \( x \) and \( y \), then \( A \) is an alternative algebra. Similar results for strongly alternative algebras, noncommutative Jordan algebras, and power-associative algebras are established.

As usual \( (x, y, z) \) denotes \( (xy)z - x(yz) \) and \( [x, y] \) denotes \( xy - yx \). Wherever convenient we will write \( a \equiv 0 \mod F \) instead of \( a \in F \cdot 1 \). Also, throughout we shall use the term "algebra" to mean a not necessarily associative algebra with identity element \( 1 \) over a field \( F \).

Our results depend on the Teichmüller or 5-identity:
\[
x(y, z, w) + (x, y, z)w = (xy, z, w) - (x, yz, w) + (x, y, zw)
\]
which holds in any nonassociative ring. We also rely heavily on the ability to linearize identities [3], [5] and on the linear independence of various elements of our algebra. Thus, we restrict our attention to algebras over fields.

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2. The alternative case. Recall that an algebra is called flexible if
\((x, y, x) = 0\) for all elements \(x, y\).

**Lemma 1.** If \(A\) is an algebra in which \((x, x^2, x) \equiv 0 \mod F\) for all \(x, y\) in \(A\), then \(2(x, x, x) = 0\) for all \(x\) in \(A\).

**Proof.** The result is trivially true if \(x \in F \cdot 1\). Assume now that \(x \notin F \cdot 1\). In \((x, x^2, x) \equiv 0 \mod F\), replace \(x\) by \(x + 1\) to yield \(2(x, x, x) \equiv 0 \mod F\). Next, linearize the relation \(2(x, x, x) \equiv 0 \mod F\) to get \(2(x^2, x, x) + 2(x, x^2, x) + 2(x, x, x^2) \equiv 0 \mod F\). Thus \(2(x^2, x, x) + 2(x, x, x^2) \equiv 0 \mod F\). By the 5-identity

\[
x(x, x, x) + (x, x, x)x = (x^2, x, x) - (x, x^2, x) + (x, x, x^2).
\]

Since \(2(x, x, x) \in F \cdot 1\) we have \(4x(x, x, x) \in F \cdot 1\). But \(x \notin F \cdot 1\). Therefore \(4(x, x, x) = 0\) and so \(2(x, x, x) = 0\).

**Lemma 2.** If \(A\) is an algebra in which \((x, x, y) \equiv (y, x, x) \equiv 0 \mod F\) for all \(x, y\) in \(A\), then \(A\) is flexible.

**Proof.** Expand \((x + y, x + y, x) \in F \cdot 1\) to get \((x, y, x) \in F \cdot 1\). Thus Lemma 1 applies.

By the 5-identity we have

\[\begin{align*}
(1) \quad (x^2, x, y) - (x, x^2, y) + (x, x, xy) - (x, x, x)y &= x(x, x, y) \\
(2) \quad (y, x, x^2) - (y, x^2, x) + (yx, x, x) - y(x, x, x) &= (y, x, x)x.
\end{align*}\]

Now add equations (1) and (2). Since \((x, y, x) \in F \cdot 1\) and \(2(x, x, x) = 0\) it follows that the left side of the resulting equation is in \(F \cdot 1\). Therefore \(x(x, x, y) + (y, x, x)x \in F \cdot 1 \cap F \cdot x\). Thus we have

\[\begin{align*}
(3) \quad (x, x, y) + (y, x, x) &= 0.
\end{align*}\]

From the 5-identity again, we get

\[\begin{align*}
(4) \quad (x^2, y, x) &= x(x, y, x) + (x, x, y)x + (x, xy, x) - (x, x, xy).
\end{align*}\]

Therefore \((x^2, y, x) \in F \cdot 1 + F \cdot x\). This gives \((x^2, x, y) \in F \cdot 1 + F \cdot x\) and \((x, x^2, y) \in F \cdot 1 + F \cdot x\). From (1) we can now conclude that \((x, x, x)y \in F \cdot 1 + F \cdot x\) for all \(x, y\) in \(A\). Thus \((x, x, x) = 0\) for all \(x\) in \(A\). For if \(\dim A > 2\), an element \(y\) can be chosen such that \(y \notin F \cdot 1 + F \cdot x\) whereas if \(\dim A \leq 2\), it is automatic that \(A\) is associative.
Now if $F$ contains more than two elements, we can linearize $(x, x, x) = 0$ to obtain $(x, x, y) + (y, x, x) + (x, y, x) = 0$ and from (3) it follows that $A$ is flexible. If $F$ contains only two elements, we imbed $F$ in a larger field $\overline{F}$ and consider the scalar extension $\overline{A} = A \otimes_F \overline{F}$. Since $(x, x, y) \in F \cdot 1$ and $(y, x, x) \in F \cdot 1$ for all $x, y$ in $F$, it is immediate that $(\overline{x}, \overline{x}, \overline{y}) \in \overline{F} \cdot 1$ for all $\overline{x}, \overline{y}$ in $\overline{A}$. Consequently $\overline{A}$ is flexible and thus $A$ is flexible also. □

San Soucie [6] has defined a strongly right alternative algebra to be a right alternative algebra $((y, x, x) = 0)$ satisfying the identity $((xy)z)y = x((yz)y)$. Thedy [9] has shown that for a right alternative algebra this is equivalent to $(y, x, x^2) = 0$ in all extensions. We now prove

**Lemma 3.** If $A$ is an algebra in which $(y, x, x) \equiv (y, x, x^2) \equiv 0 \mod F$ for all $x, y$ in $A$, then $(y, x, x) = (y, x, x^2) = 0$ for all $x, y$ in $A$.

**Proof.** From the 5-identity we have $y(x, x, x) + (y, x, x)x = (yx, x, x) - (y, x, x)x + (y, x, x^2)$ and the right side is in $F \cdot 1$ by hypothesis. Therefore we have

$$y(x, x, x) + (y, x, x)x \equiv 0 \mod F \cdot 1 \quad \text{for all } x, y \text{ in } A. \quad (5)$$

Now we may assume that there exists a $y$ in $A$ which is linearly independent of 1 and $x$ for otherwise $A$ would be associative. Thus $(x, x, x) = 0$ for all $x$ in $A$. Consequently $(y, x, x)x \in F \cdot 1 \cap F \cdot x$ for all $y$ in $A$ from which we conclude that $(y, x, x) = 0$.

Irrespective of the number of elements in $F$ we may linearize the identity $(y, x^2, x) \equiv 0 \mod F$ to get $2(y, x^3, x) \equiv 0 \mod F$. Therefore the right side of

$$y(x, x^2, x) + (y, x, x^2)x = (yx, x^2, x) - (y, x^3, x) + (y, x, x^3)$$

is in $F \cdot 1$ so that we have $(y, x, x^2) + (y, x, x^2)x \equiv 0 \mod F$ for all $x, y$ in $A$ and by the same argument as before we arrive at $(y, x, x^2) = 0$.

We remark that if $F$ has at least three elements and $(y, x, x) \equiv (y, x, x^2) \equiv 0 \mod F$, then the algebra is strongly right alternative. For by the lemma, $(y, x, x) = (y, x, x^2) = 0$ for all $x, y$ in $A$, and since $F$ has at least three elements, these identities hold in all extensions.

We are now able to prove our first main result.

**Theorem 1.** If $A$ is an algebra in which $(x, x, y) \equiv (y, x, x) \equiv 0 \mod F$ for all $x, y$ in $F$ and if the characteristic of $F \neq 3$, then $A$ is an alternative algebra.
Proof. From the 5-identity and Lemma 2 we have $x(y, x, x) \equiv (x, y, x^2) \mod F$ and $x(x, x, y) \equiv (x^2, x, y) - (x, x^2, y) \mod F$. We add these congruences to get

$$x[(y, x, x) + (x, x, y)] \equiv 3(x, y, x^2) \mod F$$

or, again by Lemma 2, $3(x, y, x^2) \equiv 0 \mod F$. Since characteristic $F \neq 3$, we have $(x, y, x^2) \equiv 0 \mod F$. Then from Lemma 3 it follows that $(y, x, x) = 0$. Linearization of the flexible law gives $(x, x, y) = 0$. Therefore $A$ is alternative.

3. The Jordan case. Recall that a ring $R$ is called noncommutative Jordan if it is flexible and satisfies the Jordan law $(x, y, x) = 0$. If $R$ is a ring in which to each $a \in R$ there is a unique $b \in R$ such that $2b = a$, we can define the attached ring $R^+$ to be the same additive group as $R$ with multiplication in $R^+$ given by $x \cdot y = 1/2(xy + yx)$ where $xy$ denotes the multiplication in $R$. In particular this applies to an algebra over a field of characteristic $\neq 2$.

Theorem 2. If $A$ is an algebra over a field $F$ of characteristic $\neq 2$ in which $[x, y] = (x, y, x) = 0 \mod F$ for all $x, y$ in $A$, then $A$ is a Jordan algebra.

Proof. Since $A$ contains an identity element 1, linearization of $(x, y, x^2) \equiv 0 \mod F$ yields $(x, y, x) \equiv 0 \mod F$ [8]. On the other hand

$$[xy, x] = x[y, x] + (x, y, x)$$

holds in any ring. Therefore we get $x[y, x] \in F \cdot 1 \cap F \cdot x$ so that we can conclude that $[x, y] = 0$ and $A$ is commutative. Hence $A$ is flexible and

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad \text{for all } x, y, z \text{ in } A.$$  

Thus $(x^2, y^2, x) = -(y^2, x, x^2) - (x, x^2, y^2)$. Linearization of the hypothesis gives $(x^2, y, z) + 2(xz, y, x) = 0 \mod F$ and thus the right side of the last equation is congruent to $2(x^2y, x, y) + 2(y, x^2, xy) \mod F$. Thus we have

$$(x^2, y^2, x) = \frac{1}{2}(x^2y, x, y) + 2(y, x^2, xy) \mod F \quad \text{for all } x, y \text{ in } A.$$  

Now by (6) again we have

$$2(x^2y, x, y) + 2(y, x^2, xy) = 2(x^2y, x, y) - 2(x^2, yx, y) + 2(x^2, y, xy).$$

Therefore, we have
(8) \( (x^2, y^2, x) \equiv 2(x^2y, x, y) - 2(x^2, yx, y) + 2(x^2, y, xy) \mod F \)

for all \( x, y \) in \( A \).

Thus, by the 5-identity and flexibility we arrive at: \( (x^2, y^2, x) \equiv 2(x^2, y, x)y \mod F \). Hence \( 2(x^2, y, x)y \in F \cdot 1 \cap F \cdot y \) for all \( x, y \) in \( A \). It follows that \( (x^2, y, x) = 0 \) and \( A \) is a Jordan algebra.

An algebra \( R \) is called Jordan admissible if \( R^+ \) is a Jordan algebra.

Corollary 1. If \( A \) is an algebra over a field \( F \) of characteristic \( \neq 2 \) in which \( (x, y, x^2) \equiv 0 \mod F \) for all \( x, y \) in \( A \), then \( A \) is a Jordan admissible algebra.

Proof. Again it follows from the hypothesis that \( (x, y, x) \equiv 0 \mod F \). Therefore Lemma 1 yields \( (x, x, x) = 0 \) and its linearized version \( (x, x, y) + (x, y, x) + (y, x, x) = 0 \). Now it is straightforward that

\[
4[(x \cdot y) \cdot x^2 - x \cdot (y \cdot x^2)] = (x, y, x^2) - (x^2, y, x) + (y, x, x^2) - (x, x, y) + (x, x^2, y) - (y, x^2, x) \mod F
\]

Hence

\[
4[(x \cdot y) \cdot x^2 - x \cdot (y \cdot x^2)] = 2(y, x, x^2) - 2(y, x^2, x) \mod F
\]

(since \( (x, y, x) \in F \cdot 1 \))

\[
= -2(yx, x, x) + 2(y, x, x)x \mod F \quad \text{(5-identity)}
\]

\[
= 2(x, x, yx) + 2(y, x, x)x \mod F
\]

\[
= 2x(x, y, x) + 2(x, y, x)x + 2(y, x, x)x \mod F \quad \text{(5-identity)}
\]

\[
= 2[(x, y, x) + (x, x, y) + (y, x, x)]x \mod F
\]

= 0

since \( (x, y, x) + (x, x, y) + (y, x, x) = 0 \). Therefore \( (x \cdot y) \cdot x^2 - x \cdot (y \cdot x^2) \equiv 0 \mod F \). Hence \( A^+ \) satisfies the conditions of Theorem 2 and is a Jordan algebra. Therefore \( A \) is Jordan admissible.

Since a flexible, Jordan admissible algebra is noncommutative Jordan [7], the following corollary is immediate.

Corollary 2. If \( A \) is a flexible algebra over a field \( F \) of characteristic \( \neq 2 \) in which \( (x, y, x^2) \equiv 0 \mod F \) for all \( x, y \) in \( A \), then \( A \) is a noncommutative Jordan algebra.

Examples. 1. The following example shows that the result of Corollary
2 is not true if the algebra is not assumed to be flexible. Let $A$ be the 4-dimensional algebra over a field $F$ of characteristic $\neq 2$ with basis $1, a, b, c$. Define multiplication by $a^2 = b^2 = c^2 = 1$, $ab = -ba = c$, and all other products zero. Then, for all $x, y$ in $A$ one notes that $(x, y, x) \equiv (x^2, y, x) \equiv 0 \mod F$. However $(a, b, c) + (c, b, a) = 2 \neq 0$. Therefore $A$ is not flexible. In addition it is easy to see that $A$ is a simple algebra.

2. There are many examples of simple, power-associative algebras in which $[x, y] \equiv 0 \mod F$ but which are not commutative. See, for example, Example 2 of [4] and the class of algebras constructed in [1].

3. The following is an example of an algebra $A$ with an idempotent $e$ in the center $C$ of $A$ such that $(x, y, z) \in F \cdot e \subseteq C$ for all $x, y, z$ in $A$, but $A$ is not even power-associative. Let $A$ be the 4-dimensional algebra with basis $e, x, y, z$ and multiplication given by: $xy = yx = z, e^2 = zx = xz = yz = zy = e$ and all other products zero. Thus, the results of Theorem 1 would be false if the congruences were assumed modulo the center.

4. The power-associative case. In an arbitrary algebra $A$ powers of elements $x$ in $A$ are defined inductively by $x^n = xx^{n-1}$. $A$ is called power-associative if $x^m x^n = x^{m+n}$ for all $x$ in $A$ and for all positive integers $m, n$. This is easily equivalent to $(x^p, x^q, x^r) = 0$ for all $x, p, q, r$.

**Theorem 3.** If $A$ is an algebra in which $(x^p, x^q, x^r) \equiv 0 \mod F$ for all $x$ in $A$ and all positive integers $p, q, r$, then $A$ is a power-associative algebra.

**Proof.** Let $x$ be in $A$. Then by the 5-identity we have

$$x(x, x, x^2) + (x, x, x)x^2 = (x^2, x, x^2) - (x, x^2, x^2) + (x, x, x^3).$$

By hypothesis the right side is in $F \cdot 1$. Thus $x(x, x, x^2) + (x, x, x)x^2 \equiv 0 \mod F$. It follows that either $(x, x, x) = 0$ or $x^2 \in F \cdot 1 + F \cdot x$. But the latter also implies that $(x, x, x) = 0$. Assume inductively that $x^m x^n = x^{m+n}$ for $3 \leq m + n \leq N$. Now let $m + n = N + 1$. If $m = 1$, then $x^m x^n = x^{m+n}$ by definition. If $m > 1$, then by the induction hypothesis we have

$$x^m x^n = (xx^{m-1})x^n = (x, x^{m-1}, x^n).$$

Thus, if $(x, x^{m-1}, x^n) = 0$ we are done. By the 5-identity we have $x(x^{m-1}, x^n, x^2) + (x, x^{m-1}, x^n)x^2 \equiv 0 \mod F$. Now if $(x, x^{m-1}, x^n) \neq 0$, then $x^2 \in F \cdot 1 + F \cdot x$ which implies that $x^n \in F \cdot 1 + F \cdot x$ and $x^{m-1} \in F \cdot 1 + F \cdot x$. Thus, $(x, x^{m-1}, x^n) = 0$ (since $(x, x, x) = 0$). Therefore $x^m x^n = x^{N+1} = x^{m+n}$ and $A$ is power-associative.
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