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THE RADIUS OF α-CONVEXITY FOR THE CLASS OF STARLIKE UNIVALENT FUNCTIONS, α-REAL

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ABSTRACT. We use a result due to Gutljanskiĭ to obtain the radius of α-convexity for the class S^* of starlike univalent functions for real α.

1. **Introduction.** Let F denote a nonempty collection of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{1}$$

each of which is univalent in the unit disc $\Delta \equiv \{z \mid |z| < 1\}$ and let

$$J(\alpha, f(z)) \equiv \operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}, \tag{2}$$

where α is a given real number. Then the real number

$$R_\alpha(F) \equiv \sup\{R \mid J(\alpha, f(z)) > 0,\ |z| < R,\ f \in F\} \tag{3}$$

is called the *radius of α-convexity* of F.

Let S^* be the class of functions (1) which are starlike in Δ, i.e. which verify the condition

$$\operatorname{Re}(zf'(z)/f(z)) > 0, \qquad z \in \Delta. \tag{4}$$

In this paper we obtain $R_\alpha(S^*)$ for all real α. This result, announced in [4], improves an earlier one obtained by S. S. Miller and us [2]. The three of us found the value of $R_\alpha(S^*)$ for $\alpha \geq 0$ by using well-known elementary estimates for the functionals $\operatorname{Re} p(z)$ and $\operatorname{Re}(zp'(z)/p(z))$ for analytic functions $p(z) = 1 + \cdots$ whose real part is positive in Δ. Indeed, it is Gutljanskiĭ's complicated and sophisticated relations between $\operatorname{Re} p(z)$ and $\operatorname{Re}[p(z) + zp'(z)/p(z)]$ [1] that enable us to obtain a complete solution to the problem of determining the radius $R_\alpha(S^*)$ for all real α.

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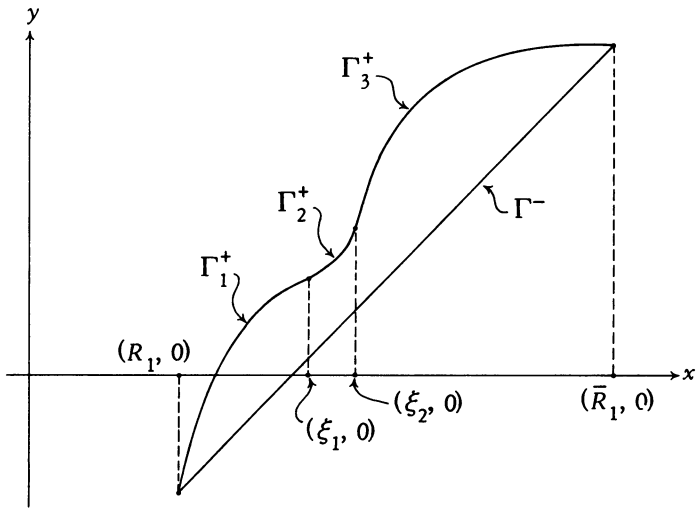
Here is Gutljanskiĭ's result, as he stated it, along with a diagram that will help in following our application of this result.

Lemma (Gutljanskiĭ [1]). *The domain of variability D of the functional*

$$\mathfrak{J}(f) \equiv \operatorname{Re}(zf'(z)/f(z)) + i \operatorname{Re}(1 + (zf''(z)/f'(z)))$$

within the class S^*, for fixed $z \in \Delta$, is bounded by two curves Γ^+ and Γ^- as shown in the diagram below.



Γ^+ consists of three arcs $\Gamma_1^+, \Gamma_2^+, \Gamma_3^+$, where

$$\Gamma_1^+ : y = \Phi_1(x) \equiv \tfrac{1}{2}x[3 - (2ax - 1)^{-1}], \qquad R_1 \le x \le \xi_1,$$

where $a \equiv (1 + r^2)/(1 - r^2)$, $|z| \equiv r$, $R_1 \equiv (1 - r)/(1 + r)$ and ξ_1 is the unique positive root of the equation $(2ax - 1)^{3/2} = x$;

$$\Gamma_2^+ : y = \Phi_2(x) \equiv \frac{3}{2}x - \frac{y_0^3 - (2ax - 1)y_0 + x}{2y_0^2},$$

for $\xi_1 \le x \le \xi_2 \equiv [(a^2 + 3)^{1/2} - a]$, where y_0 is the unique positive root of $y^3 + (2ax - 1)y - 2x = 0$; and

$$\Gamma_3^+ : y = \Phi_3(x) \equiv x - x^{-1} + a, \qquad \xi_2 \le x \le \bar{R}_1 \equiv (1 + r)/(1 - r).$$

The arc Γ^- is a line segment.

$$\Gamma^- : y = 2x - a, \qquad R_1 \le x \le \bar{R}_1.$$

It is easy to show that Γ_1^+ and Γ_3^+ are concave curves, while Γ_2^+ is convex, relative to the horizontal axis.

2. Main result.

Theorem. *The radius of α-convexity of the class S^* of all starlike functions is given by*

$$R_\alpha(S^*) = \begin{cases} (1+\alpha) - [(1+\alpha)^2 - 1]^{1/2}, & \alpha \geq 0, \\ \left[\dfrac{2-(-\alpha)^{1/2}}{2+(-\alpha)^{1/2}}\right]^{1/2}, & -3 \leq \alpha \leq 0, \\ -(1+\alpha) - [(1+\alpha)^2 - 1]^{1/2}, & \alpha \leq -3. \end{cases}$$

Proof. For fixed α, z, we consider the line

$$L_\alpha\colon (1-\alpha)x + \alpha y = 0 \quad \text{or} \quad y = \frac{\alpha - 1}{\alpha}x.$$

Now if $f \in S^*$ and if z is fixed, then $J(\alpha, f(z)) \equiv (1-\alpha)x + \alpha y$, where $x \equiv \operatorname{Re}(zf'(z)/f(z))$, $y \equiv \operatorname{Re}(1 + zf''(z)/f'(z))$ for some $(x, y) \in D$. Hence the problem of finding $R_\alpha(S^*)$ is replaced by the problem of finding when L_α is a *support line from below* (above) of the domain of variability D for α positive (negative).

(i) *Suppose* $\alpha > 0$. In this case we want D to be supported from *below* by L_α. Since the slope of L_α is never greater than one, and since Γ^- has slope two, it follows that L_α supports D when L_α passes through $(R_1, \Phi_1(R_1))$, that is, when the equation

$$(1-\alpha)\frac{1-r}{1+r} + \alpha\frac{1-4r+r^2}{1-r^2} = 0 \tag{5}$$

holds. From (5) we obtain

$$R_\alpha(S^*) = (1+\alpha) - [(1+\alpha)^2 - 1]^{1/2},$$

a result obtained earlier by S. S. Miller and us [3].

(ii) *Let* $\alpha < 0$. In this case we want L_α to be a line of support for D from *above*, that is, D lies below L_α except for the points of Γ^+ that lie on L_α. It is (geometrically) clear that L_α is such a support line if and only if L_α passes through $(R_1, \Phi_1(R_1))$, or L_α is tangent to Γ_1^+, or L_α is tangent to Γ_3^+, or L_α passes through $(\bar{R}_1, \Phi_3(R_1))$. Here we have used the fact that Γ_2^+ "opens up" so that L_α *cannot* be a support line for D and also be tangent to Γ_2^+ for some $\bar{x}$, $\xi_1 < \bar{x} < \xi_2$,

(ii–a) If L_α passes through $(R_1, \Phi_1(R_1))$, then $(1-\alpha)R_1 + \alpha\Phi_1(R_1) = 0$, and hence $1 - 2(1+\alpha)r + r^2 = 0$. But this last implies that $r + 1/r = 2(1+\alpha)$, which (for $\alpha < 0$) is not compatible with the inequality $r + 1/r \geq 2$ for all $r > 0$. Hence L_α cannot pass through $(R_1, \Phi_1(R_1))$.

(ii–b). *Suppose* $\alpha \leq -1$. We shall show that when L_α is a support line (from above) for D, then either L_α passes through $(\bar{R}_1, \Phi_3(\bar{R}_1))$ and/or L_α is tangent to Γ_3^+.

First, since Γ_2^+ "opens up", L_α cannot intersect Γ_2^+ and still be a support line for D. Second, the slope of L_α, $1 - 1/\alpha$ satisfies the inequalities $1 \leq 1 - 1/\alpha \leq 2$ in this case. Third, since the slope function $\Phi_1'(x)$ of Γ_1^+ is a decreasing function on the interval $R_1 \leq x \leq \xi_1$, the minimum of $\Phi_1'(x)$ on that interval is $\Phi_1'(\xi_1)$. Elementary calculations show $\Phi_1'(\xi_1) = \frac{1}{2}[3 + \xi_1^{-4/3}]$; here we have used the relation $(2\alpha\xi_1 - 1)^{2/3} = \xi_1$. But for $0 < \xi_1 < 1$ we see that $\Phi_1'(\xi_1) > 2$, and hence $\Phi_1'(\xi_1)$ dominates the slope of L_α. Therefore L_α, if it is a supporting line of D, *cannot* be tangent to Γ_1^+.

We now know that if L_α is a supporting line of D, then L_α must pass through the point $(\bar{R}_1, \Phi_3(\bar{R}_1))$ and/or be tangent to Γ_3^+.

Suppose L_α is tangent to Γ_3^+ and lies above D. Then there exists x_2 such that $(\alpha - 1)/\alpha = 1 + 1/x_2^2 = \Phi_3'(x_2)$, which shows $x_2 = (-\alpha)^{1/2}$. Since Γ_3^+ is concave, we have $\xi_2 \leq x_2 \leq \bar{R}_1 = (1+r)/(1-r)$. Since D lies below L_α and since L_α is tangent to Γ_3^+ at $(x_2, \Phi_3(x_2))$, we conclude that for each $f \in S^*$, and for each fixed z, we have

$$\begin{aligned} J(\alpha, f(z)) &\geq (1-\alpha)x_2 + \alpha\Phi_3(x_2) \\ &\geq (1-\alpha)(-\alpha)^{1/2} + \alpha[(-\alpha)^{1/2} - (-\alpha)^{-1/2} + a] = 0, \end{aligned} \tag{6}$$

where $a = (1+r^2)/(1-r^2)$. From (6) we solve for r to obtain

$$r_2 = [(2 - (-\alpha)^{1/2})/(2 + (-\alpha)^{1/2})]^{1/2},$$

which must satisfy the necessary condition $x_2 = (-\alpha)^{1/2} \leq \bar{R}_1 = (1+r_2)/(1-r_2)$. But this last holds only when $-3 \leq \alpha$. Hence we have found $R_\alpha(S^*) = r_2$, but only for the range $-3 \leq \alpha \leq 1$.

Now we have the remaining case, a study of the supporting line for the case $\alpha < -3$. In this case L_α passes through the point $(\bar{R}_1, \Phi_3(\bar{R}_1))$. Hence for each $f \in S^*$ and for each fixed z, we have

$$J(\alpha, f(z)) \geq (1-\alpha)\bar{R}_1 + \alpha\Phi_3(\bar{R}_1) \geq (1-\alpha)\left(\frac{1+r}{1-r}\right) + \alpha\frac{1+4r+r^2}{1-r^2} = 0,$$

which yields the solution

$$r_3 = -(1+\alpha) - [(1+\alpha)^2 - 1]^{1/2},$$

that is, $R_\alpha(S^*) = r_3$ for the case $\alpha < -3$.

(iii) *Suppose* $-1 \le \alpha < 0$. If L_α is tangent to Γ_1^+ at some point $(x_1, \Phi_1(x_1))$, $R_1 \le x_1 \le \xi_1$, then

$$\Phi_1'(x_1) = \frac{3}{2}\left[1 + \frac{1}{3}\frac{1}{(2ax_1 - 1)^2}\right] = \frac{\alpha - 1}{\alpha}$$

must hold, that is

$$2ax_1 = 1 + [-(\alpha+2)/\alpha]^{1/2}.$$

Since $(x_1, \Phi_1(x_1))$ lies on L_α, we have

$$x_1\left[1 + \frac{\alpha}{2} - \frac{\alpha}{2}\left(-\frac{2+\alpha}{\alpha}\right)^{1/2}\right] = (1-\alpha)x_1 + \alpha\Phi_1(x_1) = 0$$

must hold. But for $-1 \le \alpha < 0$, this last relation cannot hold. Hence if L_α is a support line for D, then L_α is *not* tangent to Γ_1^+.

If L_α is a support line for D and if L_α passes through $(R_1, \Phi_1(R_1))$, then (5) holds. But the left-hand member of (5) is always positive for $-1 \le \alpha < 0$, for $0 \le r < 1$. Hence if L_α supports D from above, for $-1 \le \alpha < 0$, then L_α either passes through $(\bar{R}_1, \Phi_3(\bar{R}_1))$ and/or is tangent to Γ_3^+. But we have shown that $\alpha \le -3$ is a necessary condition for L_α to support D and to pass through $(\bar{R}_1, \Phi_3(\bar{R}_1))$. We conclude that in the present case, $-1 \le \alpha < 0$, when L_α is a line of support for D, then L_α is tangent to Γ_3^+. This leads to $R_\alpha(S^*) = r_2$ as in (ii-b) above.

Since it is (geometrically) clear that for each real α, for each fixed z, there is the appropriate supporting L_α, out proof is now complete.

3. **Application.** Let S denote the set of all functions (1) that are univalent in Δ, and let Σ denote the set of all functions of the form

$$\phi(\zeta) \equiv 1/f(1/\zeta) = \zeta + b_0 + b_1/\zeta + \cdots \tag{7}$$

analytic, univalent and nonvanishing for $|\zeta| > 1$. Since it is now well-known that the set

$$\mathfrak{M}_\alpha \equiv \{f \mid f \in S,\ J(\alpha, f(z)) > 0,\ |z| < 1\}$$

is a subset of the set of starlike S^* $(\equiv \mathfrak{M}_0)$ [3], it follows that the set

$$\Sigma_\alpha \equiv \{\phi | \phi \in \Sigma,\ J(\alpha,\ \phi(\zeta)) > 0,\ |\zeta| > 1\}$$

is a subset of the subset Σ^* of starlike elements of Σ, and $\Sigma^* \equiv \Sigma_0$. Moreover, a calculation shows that if the relation (7) between $f \in S$ and $\phi \in \Sigma$ is written as $\phi = T(f)$, then $T(S) \equiv \Sigma$ and

$$J(\alpha,\ f(z)) = J(-\alpha,\ T(f)) \tag{8}$$

so that $\Sigma_{-\alpha} \equiv T(S_\alpha)$.

If Φ is a nonempty subset of Σ, then the real number

$$\mathrm{P}_\alpha(\Phi) \equiv \inf[\rho | J(\alpha,\ \phi(\zeta)) > 0,\ \phi \in \Phi,\ |\zeta| > \rho]$$

can be called the radius of α-convexity of Φ. If we define F such that $\Phi \equiv T(F)$, then the following formulas follow from (8):

$$R_\alpha(F) = 1/\mathrm{P}_{-\alpha}(\Phi), \qquad \mathrm{P}_\alpha(\Phi) = 1/R_{-\alpha}(F).$$

In particular $\mathrm{P}_1(\Sigma^*) = (1/R_{-1}(S^*)) = \sqrt{3}$. But $J(1, \phi(\zeta)) \equiv \mathrm{Re}(1 + \zeta\phi''(\zeta)/\phi'(\zeta))$, so that the constant $\mathrm{P}_1(\Sigma^*)$ is the radius of convexity of the set Σ^*. Thus we have recaptured a well-known result.

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