A CATEGORY THEOREM FOR TSUJI FUNCTIONS

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ABSTRACT. If $H$ denotes the functions analytic in the open unit disk with the topology of uniform convergence on compact subsets, both the Tsuji functions in $H$ and the functions in $H$ with nonempty Tsuji sets comprise sets of first category in $H$. A question is posed about the category of a class of functions containing the Tsuji functions.

1. Introduction. Let $D = \{ |z| < 1 \}$, $C = \{ |z| < 1 \}$, and $H$ be the collection of functions analytic in $D$ with the topology of uniform convergence on compact subsets of $D$. For each $f \in H$ and $z \in D$ let $f^*(z) = |f'(z)|/(1 + |f(z)|^2)$, the spherical derivative of $f$ at $z$. For each $r$, $0 < r < 1$, and each $f \in H$, we let $L(f, r) = \int_0^{2\pi} f^*(re^{i\theta})d\theta$. If $\limsup_{r \to 1} L(f, r) < \infty$, $f$ is called a Tsuji function. (First introduced in [4], the Tsuji functions have since been extensively studied [1], [2], [3].)

If, for each $\alpha \in D$, we let $\phi_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha}z)$, the Tsuji set of $f \in H$ is the set of points $\alpha \in H$ for which $f \circ \phi_\alpha$ is a Tsuji function. Tsuji sets were defined in [2], and they have not yet been characterized. In this note we prove the following

Theorem. The collection of functions in $H$ which have a nonempty Tsuji set is of first category in $H$.

This result also shows that the Tsuji functions in $H$ are of first category in $H$, which strengthens a result proved by F. Bagemihl [1].

2. Proof of the Theorem. Letting $\mathcal{T}$ be the collection of functions in $H$ having a nonempty Tsuji set, we will show that $\mathcal{T}$ is a countable union of sets which are closed and nowhere dense in $H$. If $f \in \mathcal{T}$, then for some $\alpha \in D$, $x > 0$, and $y \in (0, 1)$, $L(f \circ \phi_\alpha, r) \leq x$ for all $r \in (y, 1)$. For each triple $(n, m, k)$ of positive integers let $T(n, m, k)$ be the set of functions in $H$ for which there exists $\alpha \in D$, $|\alpha| \leq 1 - 1/n$, such that $L(f \circ \phi_\alpha, r) \leq m$ for all $r \in (1 - 1/(k + 1), 1)$. It is clear that $\mathcal{T} = \bigcup_{(n, m, k)} T(n, m, k)$, the
union being taken over the triples described.

To prove that each $T(n, m, k)$ is closed in $H$, we first state a preparatory lemma.

**Lemma 1.** Let $\{\alpha_j\}_{j=1}^{\infty}$ be a sequence in $D$ with $\alpha_j \to \alpha \in D$. Let $\phi(z) = (z - \alpha)/(1 - \bar{\alpha}z)$, and for each $j$, $\phi_j(z) = (z - \alpha_j)/(1 - \bar{\alpha}_jz)$. For a sequence $\{f_j\}_{j=1}^{\infty} \subset H$ with $f_j \to f$ in $H$:

(i) $\phi_j \to \phi$ in $H$,
(ii) $f_j \circ \phi_j \to f \circ \phi$ in $H$,
(iii) $\{(f_j \circ \phi_j)^*\}_{j=1}^{\infty}$ converges to $(f \circ \phi)^*$ uniformly on compact subsets of $D$,
(iv) for each $r \in (0, 1)$, $L(f_j \circ \phi_j, r) \to L(f \circ \phi, r)$.

**Lemma 2.** Each $T(n, m, k)$ is closed in $H$.

**Proof.** Let $\{f_j\}$ be a sequence in $T(n, m, k)$ with $f_j \to f$ in $H$. For each $j$ there is a point $\alpha_j \in D$, $|\alpha_j| \leq 1 - 1/n$, such that $L(f_j \circ \phi_j, r) \leq m$ when $r \in (1 - 1/(k + 1), 1)$. We may suppose $\alpha_j \to \alpha$, where $|\alpha| \leq 1 - 1/n$, and let $\phi(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. Lemma 1(iv) shows that $L(f \circ \phi, r) \leq m$ for each $r \in (1 - 1/(k + 1), 1)$, so that $f \in T(n, m, k)$.

**Lemma 3.** Each $T(n, m, k)$ is nowhere dense in $H$.

**Proof.** For an arbitrary $f \in T(n, m, k)$ we shall show there exists a sequence in $H - T(n, m, k)$ which converges in $H$ to $f$. Since $T(n, m, k)$ is closed, this will show it is nowhere dense in $H$.

For some $\alpha \in D$, $|\alpha| \leq 1 - 1/n$, $L(f \circ \phi_\alpha, r) \leq m$ for all $r \in (1 - 1/(k + 1), 1)$. For each positive integer $q$ let $S_q$ be the $q$th partial sum of the Maclaurin's series for $f$.

Given $q$, let $p(q)$ be a positive integer, and define $g_q(z) = S_q(z) + z^p(q)$. As long as $|p(q)|$ is increasing, $g_q \to f$ in $H$. If $p(q)$ is sufficiently large, on $C$ both $|g_q'| > (q + p(q))/2$ and $|g_q| \leq |S_q| + 1$. Thus we may take $p(q) \in (0, 1)$ so that every Jordan curve in the annulus $\rho(q) < |z| < 1$ whose interior contains $0$ is mapped by $g_q$ onto a closed curve of spherical length at least $\rho(q)$. If $p(q)$ is sufficiently large and $\rho(q)$ is near enough to $1$, we will have $L(g_q \circ \phi_\alpha, r) > m$ for a value of $r > 1 - 1/(k + 1)$.

With suitable choice of the sequence $\{p(q)\}_{q=1}^{\infty}$, the sequence $\{g_q\}_{q=1}^{\infty}$ lies in $H - T(n, m, k)$ and converges to $f$ in $H$.

3. In his paper on Tsuji functions [3], W. K. Hayman introduces a larger related class of functions. A function $f \in H$ lies in class $T_2$ if
there exists a sequence \( \{ J_n \}_{n=1}^{\infty} \) of Jordan curves in \( D \) such that: (i) \( J_n \subseteq \text{int} J_{n+1} \); (ii) \( \min J_n |z| \rightarrow 1 \) as \( n \rightarrow \infty \); (iii) \( \limsup_{n \rightarrow \infty} \int_{J_n} f^*(z) |dz| < \infty \).

The class \( T_2 \) contains the functions in \( H \) with nonempty Tsuji set, so the following question is natural.

**Question.** Is the class \( T_2 \) of first category in \( H \)?

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REFERENCES


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