INTERSECTING UNIONS OF CONVEX SETS IN $\mathbb{R}^n$

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ABSTRACT. Let $\mathcal{C} = \{C_a : a \text{ in some index set } I\}$ be a collection of convex sets, and let $\mathfrak{M} = \{C_a \cup C_{\beta} : a \neq \beta, C_a, C_\beta \text{ in } \mathcal{C}\}$. In this paper, various decomposition theorems are obtained for the set $\bigcap \mathfrak{M}$.

1. Introduction. In [1], it is proved that if $\mathcal{C}$ is a collection of closed convex sets in the plane and if $\mathfrak{M} = \{A \cup B : A, B \text{ distinct members of } \mathcal{C}\}$, then the set $\bigcap \mathfrak{M}$ is expressible as a union of three or fewer closed convex sets. In this paper, an attempt is made to obtain similar decompositions without the restriction that $\mathcal{C}$ be planar. Although several theorems are stated for an arbitrary linear topological space, restrictions on the convex sets reduce the setting to $\mathbb{R}^n$, and all the theorems are essentially finite dimensional ones. Throughout the paper, $\text{aff } S$ and $\text{ker } S$ will be used to denote the affine hull and kernel, respectively, for the set $S$. If $S$ is convex, $\dim S$ will denote the dimension of the affine hull of $S$, and for convenience, we say that the dimension of the null set is $-1$.

2. Decomposition theorems for $\bigcap \mathfrak{M}$. The following easy lemmas will be useful.

Lemma 1. Let $\mathcal{C} = \{C_a : a \text{ in some index set } I\}$ be a collection of sets, and let $\mathfrak{M} = \{C_{a_1} \cup \ldots \cup C_{a_k} : a_1, \ldots, a_k \text{ distinct members of } I\}$. Then $x \in \bigcap \mathfrak{M}$ if and only if there are at most $k - 1$ members $a$ in $I$ for which $x \notin C_a$.

Lemma 2. Let $\mathcal{C} = \{C_a : a \text{ in some index set } I\}$ be a collection of convex sets in some linear topological space, and let $\mathfrak{M} = \{C_{a_1} \cup \ldots \cup C_{a_k} : a_1, \ldots, a_k \text{ distinct members of } I\}$. Then $\bigcap \mathcal{C} \subseteq \ker(\bigcap \mathfrak{M})$.

Theorem 1. Let $\mathcal{C} = \{C_a : a \text{ in some index set } I\}$ be a collection of convex sets in some linear topological space, and assume that, for some $n \geq 1$, at least $n + 1$ of these sets have dimension no greater than $n - 1$. 

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Letting $\mathcal{M} = \{ \mathcal{C}_\alpha \cup \mathcal{C}_\beta : \alpha \neq \beta, \mathcal{C}_\alpha, \mathcal{C}_\beta \in \mathcal{E} \}$, if $\dim \text{aff}(\bigcap \mathcal{M})$ is at least $n$, then $\bigcap \mathcal{M}$ is a union of $n + 1$ or fewer convex sets, each containing $\bigcap \mathcal{C}$. The number $n + 1$ is best possible for every $n$.

Proof. We use an inductive argument. If $n = 1$, then at least two members $A, B$ of $\mathcal{C}$ are singleton sets, $\bigcap \mathcal{M} \subseteq A \cup B$, and trivially $\bigcap \mathcal{M}$ consists of exactly two points.

Assume that the result is true for every integer $m, 1 \leq m \leq n - 1$, to prove for $n$. There are two cases to consider.

Case 1. Suppose that there are $n + 1$ affinely independent points $x_1, \ldots, x_{n+1}$ of $\bigcap \mathcal{M}$ not in $\bigcap \mathcal{C}$. Then for each $i$, $1 \leq i \leq n + 1$, we may select a corresponding set $A_i$ in $\mathcal{C}$ with $x_i \notin A_i$. For any $C$ in $\mathcal{C} \sim \{ A_1, \ldots, A_{n+1} \}$, $C$ necessarily contains each of the $n + 1$ affinely independent points $x_1, \ldots, x_{n+1}$, and so $\dim C \geq n$. Hence the $A_i$ sets must be exactly those members of $\mathcal{C}$ which have dimension no greater than $n - 1$, and the $A_i$ sets are necessarily distinct, $1 \leq i \leq n + 1$. Then each $A_i$ must contain each of the $n$ points $x_j, j \neq i, 1 \leq j \leq n + 1$. Since the points $x_1, \ldots, x_{n+1}$ are vertices of an $n$-dimensional simplex, each $A_i$ lies in the affine hull of a facet of the simplex. Therefore $A_1 \cap \ldots \cap A_{n+1} = \emptyset$ and $\bigcap \mathcal{M}$ is just the union of the $n + 1$ convex sets $B_i$, where $B_i = \bigcap \{ C : C \in \mathcal{C}, C \notin A_i \} = \{ x_i \}, 1 \leq i \leq n + 1$.

Case 2. If there are at most $k + 1 < n + 1$ affinely independent points $x_1, \ldots, x_{k+1}$ of $\bigcap \mathcal{M}$ not in $\bigcap \mathcal{C}$, these points lie in a $k$-dimensional flat $\pi$ (and clearly we may assume $0 \leq k$ for otherwise the result is trivial). Select points $x_{k+2}, \ldots, x_{n+1}$ in $\bigcap \mathcal{M}$ so that $x_1, \ldots, x_{k+1}, x_{k+2}, \ldots, x_{n+1}$ are affinely independent. Then each of the $n - k$ points $x_{k+2}, \ldots, x_{n+1}$ must lie in $\bigcap \mathcal{C}$. For each of the members $A$ of $\mathcal{C}$ for which $\dim A \leq n - 1$, there are no more than $n - (n - k) = k$ affinely independent points of $A$ in $\pi$, and $\dim(A \cap \pi) \leq k - 1$. Hence $\mathcal{C}' = \{ C \cap \pi : C \in \mathcal{C} \}$ is a collection of convex sets, at least $n + 1 > k + 1$ of which have dimension no greater than $k - 1$. Letting $\mathcal{M}' = \{ \mathcal{C}_\alpha \cup \mathcal{C}_\beta : \alpha \neq \beta, \mathcal{C}_\alpha, \mathcal{C}_\beta \in \mathcal{C}' \}$, $\dim \text{aff}(\bigcap \mathcal{M}') = \dim \text{aff}(\bigcap \mathcal{C} \cap \pi) = k$. Therefore, by our induction hypothesis, $\bigcap \mathcal{M}'$ is a union of $k + 1$ or fewer convex sets, say $S_1', \ldots, S_{k+1}'$, each containing $\bigcap \mathcal{C}'$.

We assert that $\bigcap \mathcal{M}$ is a union of the $k + 1$ convex sets $S_i = S_i' \cup (\bigcap \mathcal{C})$, $1 \leq i \leq k + 1$: For $x$ in $\bigcap \mathcal{M}$ and $x$ not in any $S_i'$, $1 \leq i \leq k + 1$, then $x \notin \pi$, so $x$ must belong to every $C$ in $\mathcal{C}$. Hence $S_1 \cup \ldots \cup S_{k+1} = \bigcap \mathcal{M}$. To show that each $S_i$ is convex, clearly we need only consider $r$ in $S_i'$, $s$ in $\bigcap \mathcal{C}$.
to show that \([s, r] \subseteq S_i\). Now by Lemma 2, \(s\) is in \(\text{ker}(\bigcap C_i)\), so \([s, r] \subseteq \bigcap C_i\). If \(s \in \pi\), the result is immediate since \(\bigcap C_i \subseteq S_i\). Otherwise, \([s, r] \cap \pi = \emptyset\), so \([s, r] \subseteq \bigcap C_i \sim \pi \subseteq \bigcap C\), and \([s, r] \subseteq (\bigcap C) \cup S_i = S_i\). Thus \(S_i\) is convex, \(1 \leq i \leq k + 1\), and the assertion is proved, finishing Case 2.

This completes the inductive argument, and we conclude that the statement of the theorem is true for every integer \(n \geq 1\).

**Remark.** To see that the bound of \(n + 1\) in Theorem 1 is best possible, refer to Example 1 of this paper.

**Theorem 2.** Let \(\mathcal{C} = \{C_\alpha: \alpha \in \text{some index set } I\}\) be a collection of convex sets in \(R^n\), \(n \geq 1\), and let \(\mathcal{M} = \{C_\alpha \cup C_\beta: \alpha \neq \beta, C_\alpha, C_\beta \in \mathcal{C}\}\). If there is an \(n + 1\) member subset \(J\) of \(I\) such that \(\text{aff}(C_\alpha \cap (\bigcap \mathcal{M})) \neq \text{aff}(C_\beta \cap (\bigcap \mathcal{M}))\) for \(\alpha \neq \beta, \alpha \in J, \beta \in I\), then \(\bigcap \mathcal{M}\) is a union of \(n + 1\) or fewer convex sets, each containing \(\bigcap \mathcal{C}\). The number \(n + 1\) is best possible.

**Proof.** The inductive argument of Theorem 1 may be suitably adapted to yield the result. The only significant difference appears in Case 2: As in Case 2, affinely independent points \(x_1, \ldots, x_{k+1}, x_{k+2}, \ldots, x_{j+1}\) are selected in \(\bigcap \mathcal{M}\) with \(x_1, \ldots, x_{k+1}\) in the \(k\)-dimensional flat \(\pi\) and not in \(\bigcap \mathcal{C}\), and \(x_{k+2}, \ldots, x_{j+1}\) in \(\bigcap \mathcal{C} \sim \pi\), where \(j = \dim \text{aff}(\bigcap \mathcal{M})\) and \(0 \leq k \leq j\). Then for \(\alpha \in J, \beta \in I, \alpha \neq \beta\),

\[
\text{aff}(C_\alpha \cap \pi \cap (\bigcap \mathcal{M})) \neq \text{aff}(C_\beta \cap \pi \cap (\bigcap \mathcal{M})),
\]

for otherwise

\[
\text{aff}([C_\alpha \cap \pi \cap (\bigcap \mathcal{M})] \cup \{x_{k+2}, \ldots, x_{j+1}\}) = \text{aff}([C_\beta \cap \pi \cap (\bigcap \mathcal{M})] \cup \{x_{k+2}, \ldots, x_{j+1}\}),
\]

and since \(x_{k+2}, \ldots, x_{j+1}\) are in every \(C\) in \(\mathcal{C}\),

\[
\text{aff}(C_\alpha \cap (\bigcap \mathcal{M})) = \text{aff}(C_\beta \cap (\bigcap \mathcal{M})),
\]

clearly impossible. Hence the induction hypothesis may be applied to the sets \(\mathcal{C}'\) and \(\bigcap \mathcal{M}'\) of Case 2 to complete the argument.

The following example shows that the bound of \(n + 1\) in Theorems 1 and 2 is best possible.

**Example 1.** For \(n \geq 1\), let \(T\) denote an \(n\)-dimensional simplex and \(\mathcal{C}\) the collection of facets of \(T\). Then \(\mathcal{C}\) has \(n + 1\) members, \(\bigcap \mathcal{C} = \emptyset\), and \(\bigcap \mathcal{M}\) is the collection of points which lie in exactly \(n\) facets of \(T\). Hence
\( \cap M \) is just the vertex set of \( T \) and consists of \( n + 1 \) isolated points.

Another kind of decomposition is given in Theorem 3.

**Theorem 3.** Let \( C = \{ C_\alpha : \alpha \text{ in some index set } I \} \) be a collection of closed convex sets, and let \( M = \{ C_\alpha \cup C_\beta : \alpha \neq \beta, C_\alpha, C_\beta \text{ in } C \} \). If for some \( k \geq 1 \) members \( \alpha_1, \ldots, \alpha_k \) in \( I \), \( \dim (C_{\alpha_1} \cap \cdots \cap C_{\alpha_k}) \leq i \), \( -1 \leq i \leq 2 \), then \( \cap M \) is a union of \( k + i + 1 \) or fewer closed convex sets. The bound is best possible for every pair \( k, i \).

**Proof.** For convenience of notation, let \( C_{\alpha_i} = C_i, 1 \leq i \leq k \), and define \( D_i = \bigcap \{ C : C \in C, C \neq C_i \} \). For \( x \) in \( \bigcap M \), either \( x \) lies in one of the closed convex sets \( D_i, 1 \leq i \leq k \), or \( x \in C_1 \cap \cdots \cap C_k \).

We assert that the set \( C_1 \cap \cdots \cap C_k \cap (\bigcap M) \) is expressible as a union of \( i + 1 \) or fewer closed convex sets: Define \( C' = \{ C_1' : C \in C, C \neq C_i \} \), and let \( M' = \{ C_\alpha' : \alpha \neq \beta, C_\alpha', C_\beta' \text{ in } C' \} \). Then \( C_1 \cap \cdots \cap C_k \cap (\bigcap M) \) is exactly \( \bigcap M' \). If \( i = 2 \), then \( C' \) is a collection of closed convex sets in the plane, and by suitably adapting Theorem 1 in [1], \( \bigcap M' \) is a union of three or fewer closed convex sets, the desired result. In case \( i = 1 \), techniques used in [1] may be used to show that \( \bigcap M' \) is a union of 2 or fewer closed convex sets. For \( i = 0 \) or \( i = -1 \), the result is trivial.

Therefore, \( C_1 \cap \cdots \cap C_k \cap (\bigcap M) \) is a union of \( i + 1 \) closed convex sets, and hence \( \bigcap M \) is a union of \( k + i + 1 \) or fewer closed convex sets, finishing the proof of Theorem 3.

Example 2 reveals that the bound \( k + i + 1 \) is best possible for every pair \( k, i \).

**Example 2.** For a given \( k \geq 1 \) and for \( -1 \leq i \leq 2 \), if \( k + i \geq 1 \), let \( C \) denote the \( k + i + 1 \) facets of a simplex \( T \) in \( R^{k+i} \). Then \( k \) members of \( C \) intersect in an \( i \)-dimensional set, and \( \bigcap M \), the vertex set of \( T \), is a union of \( k + i + 1 \) closed convex sets. If \( k + i = 0 \), some member of \( C \) is empty, and \( \bigcap M \) is convex.

**Corollary.** If \( C \) is a finite collection of closed convex sets in \( R^n \) and \( \dim (\bigcap C) \leq 2 \), then the corresponding set \( \bigcap M \) is a union of \( \sigma(n) + 3 \) or fewer closed convex sets, where \( \sigma(n) = \max (n + 1, 2n - 4) \).

**Proof.** By a theorem of Katchalski [2], if all \( \sigma(n) \) sets in \( C \) have at least a 3-dimensional intersection, then so does \( \bigcap C \). Hence if \( \dim (\bigcap C) \leq 2 \), there are some \( \sigma(n) \) sets in \( C \) whose intersection has dimension no more than 2. By Theorem 3, \( \bigcap M \) is a union of \( \sigma(n) + 2 + 1 \) or fewer closed convex sets.

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REFERENCES


A CHARACTERIZATION OF THE KERNEL OF A CLOSED SET

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ABSTRACT. Let $S$ be a closed subset of some linear topological space such that $\text{int} \ker S \neq \emptyset$ and $\ker S \neq S$. Let $\mathcal{C}$ denote the collection of all maximal convex subsets of $S$ and, for any fixed $k \geq 1$, let $\mathcal{K} = \{ A_1 \cup \cdots \cup A_k : A_1, \ldots, A_k \text{ distinct members of } \mathcal{C} \}$. Then $\mathcal{K} \neq \emptyset$ and $\bigcap \mathcal{K} = \ker S$.

If $\mathcal{C}$ is the collection of all maximal convex subsets of some set $S$, it is easy to show that $\bigcap \mathcal{C} = \ker S$. This paper provides an interesting and perhaps surprising analogue of this well-known result. Throughout the paper, $\text{conv } S$, $\text{int } S$, and $\ker S$ will be used to denote the convex hull, interior, and kernel, respectively, for the set $S$.

Further, we will make use of these familiar definitions: For points $x, y$ in a set $S$, we say $x$ sees $y$ via $S$ if and only if the corresponding segment $[x, y]$ lies in $S$. A subset $T$ of $S$ is said to be a visually independent subset of $S$ if and only if for every $x, y$ in $T$, $x \neq y$, $x$ does not see $y$ via $S$.

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