A LARGE CLASS OF SMALL VARIETIES
OF LATTICE-ORDERED GROUPS

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ABSTRACT. We establish the existence of a countable collection of varieties, each of which covers the abelian variety in the lattice of varieties of lattice-ordered groups.

For each positive integer \( n \), let \( \mathcal{L}_n \) denote the variety of lattice-ordered groups (hereafter, \( l \)-groups) satisfying the law \( x^ny^n = y^nx^n \). Obviously, \( \mathcal{L}_1 = \mathcal{L}_A \), the variety of abelian \( l \)-groups, and \( \mathcal{L}_n \) is contained in \( \mathcal{L}_m \) if \( n \) is a divisor of \( m \). Martinez [4] notes that the containment \( \mathcal{L}_n \subseteq \mathcal{L}_m \) is proper if \( n \) is a proper divisor of \( m \) by considering a type of example which fails to be commutative or representable in intuitively the least way: the examples are each generated by two noncommuting positive elements \( a \) and \( x \) such that distinct conjugates of \( a \) by powers of \( x \) are disjoint. Thus, one is led to suspect that because the examples are barely nonabelian, the varieties they generate might be minimally nonabelian. It will facilitate the discussion to describe these examples as subgroups of a wreath product of ordered permutation groups. (For background, definitions, etc. concerning ordered permutation groups, the reader is referred to [1], [2].)

Let \( Z \wr Z \) be the ordered wreath product of two copies of the integers, and define

\[
G_n = \{((\ldots, w(z), \ldots), \overline{w}) \in Z \wr Z | i \equiv j \pmod{n} \Rightarrow w(i) = w(j)\}.
\]

Then \( G_n \in \mathcal{L}_n \). Let \( a, x \) denote the elements of \( G_n \) given by:

\[
\overline{a} = 0, \quad a(z) = \begin{cases} 1 & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{if } z \not\equiv 0 \pmod{n}, \end{cases}
\]

\[
x = 1, \quad x(z) = 0 \quad \forall z \in Z.
\]
Then \( \frac{a^m x^m}{m} = \frac{x^m a^m}{m} \), and
\[
(a^m x^m)(z) = \begin{cases} m & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{if } z \not\equiv 0 \pmod{n}, \end{cases}
\]
\[
(x^m a^m)(z) = (x^m)(z) + (a^m)(z + m) = (a^m)(z + m)
\]
\[
= \begin{cases} m & \text{if } z + m \equiv 0 \pmod{n}, \\ 0 & \text{if } z + m \not\equiv 0 \pmod{n}. \end{cases}
\]
Hence, \( x^m a^m = a^m x^m \) iff \( m \equiv 0 \pmod{n} \), iff \( n \mid m \). Consequently, if \( n \) is a proper divisor of \( m \), then \( m \) does not divide \( n \), so \( G_m \not\in \mathcal{L}_n \). It also follows, as Martinez suggests, that if \( \mathcal{L}_n \) is contained in \( \mathcal{L}_m \), then \( n \) is a factor of \( m \) since
\[
\mathcal{L}_n \subseteq \mathcal{L}_m \Rightarrow G_n \in \mathcal{L}_m \Rightarrow a^m x^m = x^m a^m \Rightarrow n \mid m.
\]

It seems apparent that \( G_n \) is a minimal nonabelian \( l \)-group satisfying \( x^n y^n = y^n x^n \). More precisely, we shall show that if \( n \) is prime, the variety of \( l \)-groups generated by \( G_n \) is minimal with respect to containing nonabelian members, by showing that every nonabelian, subdirectly irreducible \( l \)-group in \( \mathcal{L}_n \) contains an \( l \)-subgroup \( l \)-isomorphic to \( G_n \).

**Lemma 1.** If \( C \) is a convex \( l \)-subgroup of \( G \in \mathcal{L}_n \), then \( x^{-n} C x^n = C \) for all \( x \in G \).

**Proof.** Suppose \( 1 < c \in C \). Then
\[
1 < c < c^2 < \ldots < c^n = x^{-n} c^n x^n \in x^{-n} C x^n,
\]
so \( c \in x^{-n} C x^n \), and hence \( C \subseteq x^{-n} C x^n \). Similarly, \( C \subseteq x^n C x^{-n} \), so \( C = x^{-n} C x^n \).

**Lemma 2.** If \( C \) is a convex \( l \)-subgroup of \( G \in \mathcal{L}_n \) and \( x \in G \), then the number of distinct conjugates of \( C \) of the form \( x^{-i} C x^i \) is a divisor of \( n \).

**Proof.** Let \( i \) be the smallest positive integer such that \( x^{-i} C x^i = C \). If \( i \) is not a divisor of \( n \), then there are integers \( r \) and \( s \) such that \( m + si = k \), where \( 1 \leq k < i \), and \( k \) is the greatest common divisor of \( n \) and \( i \). Then \( x^{-k} C x^k = x^{-rn+si} C x^{rn+si} = C \), contradicting the minimality of \( i \).

**Lemma 3.** Let \( LR \) denote the variety of regular \( l \)-groups. For any positive integer \( n \), \( LR \cap \mathcal{L}_n = LA \) [4, §6.4].

Holland has shown [2] that every \( l \)-group is \( l \)-isomorphic to an \( l \)-subgroup of the \( l \)-group of all order-preserving permutations of some chain,
and that an $l$-group $G$ has a transitive representation as an $l$-group of order-preserving permutations of a chain if and only if $G$ contains a \textit{representing subgroup}, i.e., a convex prime $l$-subgroup which contains no non-trivial $l$-ideal of $G$. Among those $l$-groups which have transitive representations are all subdirectly irreducible $l$-groups. These facts, together with the preceding lemma, yield the following result.

\textbf{Theorem 1.} If $m$ and $n$ are relatively prime, then $\mathfrak{L}_m \cap \mathfrak{L}_n = \mathfrak{L}_A$.

\textbf{Proof.} Let $G$ be a subdirectly irreducible member of $\mathfrak{L}_m \cap \mathfrak{L}_n$, and let $S$ be a chain on which $G$ acts transitively. Suppose $g \in G$ fixes some $x \in S$. Choose integers $r$ and $s$ such that $rm + sn = 1$. If $h$ is any member of $G$ such that $xh \neq x$, then

$$xh = xg^{mn}h^{rm+sn} = xh^{rm}g^{mn}h^{sn} = xh^{rm+sn}g^{mn} = xhg^{mn}.$$  

Therefore, $g$ fixes $xh$, and since $h$ is arbitrary and $G$ is transitive on $S$, $g = 1$. Hence, $G_x = \{1\}$, so $G$ is totally ordered, and thus regular. But $\mathfrak{L}_R \cap \mathfrak{L}_m = \mathfrak{L}_A$, so $G$ is abelian. $\mathfrak{L}_m \cap \mathfrak{L}_n$ is generated by its subdirectly irreducible members, and these are all abelian, so $\mathfrak{L}_m \cap \mathfrak{L}_n = \mathfrak{L}_A$.

\textbf{Lemma 4.} If $C$ is a representing subgroup of $G \in \mathfrak{L}_m \setminus \mathfrak{L}_A$, then there exists $1 < x \in G$ such that $x^{-1}Cx \neq C$.

\textbf{Proof.} Suppose $x^{-1}Cx = C$ for all $1 < x \in G$. Then $C$ is an $l$-ideal of $G$, so since it is also a representing subgroup, $C = \{1\}$, and $G$ is totally ordered, therefore regular. But $\mathfrak{L}_R \cap \mathfrak{L}_m = \mathfrak{L}_A$, so $G$ must be abelian. This is a contradiction; therefore, there is a positive $x$ for which $x^{-1}Cx \neq C$.

\textbf{Lemma 5.} If $C$ is a representing subgroup of $G \in \mathfrak{L}_n$ which has $n$ distinct conjugates of the form $x^{-i}Cx^i$ for some $1 < x \in G$, then $G$ contains an $l$-subgroup $l$-isomorphic to $G_n$.

\textbf{Proof.} Suppose we have $G$, $C$, $x$ with these properties. We shall find $a_0 \in G$ such that $a_0$ and $x$ correspond under the isomorphism to the two generating elements of $G_n$ denoted $a$ and $x$ in the discussion above.

For $0 \leq i \leq n - 1$, define

$$C_i = x^{-i}Cx^i, \quad D_i = \bigcap_{j \neq i} C_j.$$

Let $S$ be the chain of right cosets of $C_0$, and let $s_0$ denote the coset $C_0$ in this chain. For $0 < i < n - 1$, define $s_i = s_0 x^i$, so that $C_i$ is the stabili-
zer of $s_i$ and $D_i$ consists of all permutations of $S$ in $G$ which fix the set \{s_j | j \neq i\}. Since by hypothesis $C_0 \neq C_i$, there exists a positive permutation $h_i \in C_0 \setminus C_i$, $i = 1, \ldots, n - 1$, whence

$$1 < g_0 = h_1 h_2 \cdots h_{n-1} \in C_0 \setminus \bigcup_{i=1}^{n-1} C_i.$$ 

Define

$$d_0 = x \wedge \left( \bigwedge_{i=1}^{n-1} x^{-i} g_0^i \right).$$

Since $g_0$ fixes only $s_0$ among $s_0, \ldots, s_{n-1}$, $x^{-i} g_0^i$ fixes only $s_i$ among $s_0, \ldots, s_{n-1}$. Thus $d_0$ moves only $s_0$ among $s_0, \ldots, s_{n-1}$; $1 < d_0 \in D_0 \setminus \bigcup_{i=1}^{n-1} D_i$. Also, $d_0 < x$ since $x$ moves $s_0$. Define $d_i = x^{-i} d_0^i$, $i = 1, \ldots, n - 1$. Then $1 < d_i \in D_i \setminus \bigcup_{j \neq i} D_j$. By convexity, for any $i$, 0 $\leq i \leq n - 1$,

$$e_i = \bigvee_{k=0}^{n-1} D_k = \bigcap_{k=0}^{n-1} C_k,$$

Put $a_i = x^{-i} a_0 x^i = d_i e_i^{-1}$, $i = 1, \ldots, n - 1$. Then $x^{-n} a_i x^n = a_i$, $1 < a_i < x$, $a_i \in D_i \setminus \bigcup_{j \neq i} D_j$, $a_i \wedge a_j = 1$ if $i \neq j$, as may be routinely checked. As a consequence of the latter, $a_i a_j = a_j a_i$ $\forall i, j$.

For any $u = ((\ldots, u(x), \ldots), \bar{u}) \in G_\nu$, define

$$u\theta := a_0^{u(0)} a_1^{u(1)} \cdots a_0^{u(n-1)} x^\bar{u}.$$

It is easily verified that $\theta$ is a group homomorphism.

That $\theta$ is a lattice homomorphism follows from the fact that in $G$,

$$1 \lor \left( a_0^{u(0)} \cdots a_0^{u(n-1)} x^\bar{u} \right) = \begin{cases} 1 & \text{if } \bar{u} < 0, \\ a_0^{u(0)} \lor a_1^{u(1)} \cdots a_0^{u(n-1)} & \text{if } \bar{u} = 0, \\ a_0^{u(0)} \cdots a_0^{u(n-1)} x^\bar{u} & \text{if } \bar{u} > 0, \end{cases}$$

which we prove as follows. Since $a_i \wedge a_j = 1$ for $i \neq j$, $\Pi_{i \in T} a_i^{\alpha(i)} \wedge \Pi_{i \in T} a_i^{\alpha(i)} = 1$ where $\alpha(i) \geq 0$ for each $i$ and $T \subseteq \{0, 1, \ldots, n - 1\}$. Given $a_0^{u(0)} \cdots a_0^{u(n-1)}$, let $T = \{i | u(i) \geq 0\}$. Then

$$\left( a_0^{u(0)} \cdots a_0^{u(n-1)} \right) \lor 1 = \prod_{i \in T} a_i^{u(i)} \left( \prod_{i \not\in T} a_i^{u(i)} \lor \prod_{i \in T} a_i^{-u(i)} \right)$$

$$= \prod_{i \in T} a_i^{u(i)} \left( \prod_{i \not\in T} a_i^{-u(i)} \land \prod_{i \in T} a_i^{u(i)} \right)^{-1}$$

$$= \prod_{i \in T} a_i^{u(i)} = a_0^{u(0)} \lor a_1^{u(1)} \cdots a_0^{u(n-1)} \lor 0.$$
Next we show $a_0^m < x$ for all $m \in \mathbb{Z}$. Since $x > a_0 > 1$, this is so for $m \leq 1$.

$$a_0^m < x \iff (d_0^{-1})^m < x \iff \left( \bigwedge_{j=1}^{n-1} (1 \lor d_0^{-1}) \right)^m < x \iff \left( \biglor\left( \bigwedge_{j=1}^{n-1} d_j^{-1} \right) \right)^m < x,$$

so it suffices to show $f^m \leq x$ for $m \geq 2$, where $f = \bigwedge_{j=1}^{n-1} d_0^{-1}$. The induction hypothesis is that $f^k \leq x$ for $0 \leq k < m$. $f^m$ is an infimum of terms, one of which is $d_0^{-1} d_0^{-1} f^{m-2} d_0^{-1}$. But

$$d_0^{-1} d_0^{-1} f^{m-2} d_0^{-1} = d_0 x^{-1} n d_0^{-1} \leq x f^{m-2} d_0^{-1} x < x$$

iff

$$f^{m-2} \leq x^{-1} n d_0^{-1} x \leq x$$

since $x^{-1} n d_0^{-1} = a_0^{-1} x^n$. Therefore, $f^m \leq x$, so $a_0^m < x$ for all $m \in \mathbb{Z}$. By conjugation $a_i^m < x$ for any $i \in \mathbb{Z}$, $i = 0, 1, \ldots, n - 1$, so $x^{-1} < a_0^{u(0)} \ldots a_n^{u(n-1)} < x$ for any $u(0), \ldots, u(n-1) \in \mathbb{Z}$, and hence

$$a_0^{u(0)} \ldots a_n^{u(n-1)} x^{-1} u < a_0^{u(0)} \ldots a_n^{u(n-1)} x u$$

if $u > 0$, for any $u(0), \ldots, u(n-1) \in \mathbb{Z}$. Thus, (1) holds.

$\theta$ is 1-1 since the element $a$ of $G_n$ given by

$$\overline{a} = 0, \quad a(z) = \begin{cases} 1 & \text{if } z \equiv 0 \pmod{n}, \\ 0 & \text{if } z \not\equiv 0 \pmod{n} \end{cases}$$

belongs to the smallest nontrivial $l$-ideal of $G_n$, but $a \theta = a_0 \neq 1$. This completes the proof of the lemma.

**Theorem 2.** If $n$ is a prime, the variety of $l$-groups $[G_n]$ generated by $G_n$ covers $\mathcal{L}A$ in the lattice of varieties of $l$-groups.

**Proof.** If $G$ is a subdirectly irreducible $l$-group in $\mathcal{L}n$ but not in $\mathcal{L}A$, then by Lemma 3, $G$ contains a representing subgroup $C$ which is not an $l$-ideal of $G$. Thus, there exists $1 < x \in G$ such that $x^{-1} C x \neq C$. Since $n$ is prime, it follows from Lemma 2 that $G$ has $n$ distinct conjugates of the form $x^{-1} C x^i$, so by Lemma 5, $G$ contains an $l$-subgroup $l$-isomorphic to $G_n$. Therefore, any variety of $l$-groups contained in $\mathcal{L}n$ and which has nonabelian members contains the variety $[G_n]$. Hence, $[G_n]$ covers $\mathcal{L}A$ in the lattice of varieties of $l$-groups if $n$ is prime.


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