ON $\Lambda(p)$ SETS WITH MINIMAL CONSTANT
IN DISCRETE NONCOMMUTATIVE GROUPS

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ABSTRACT. We compute the minimal constants for infinite $\Lambda(2\pi)$ sets in discrete noncommutative groups and as a consequence we obtain an alternate proof of Leinert's theorem on $\Lambda(\infty)$ sets.

1. Introduction. Let $G$ be a discrete group. Let $l^2(G)$ denote the space of square summable complex functions on $G$ with the norm $\|f\|_{l^2} = (\sum_x |f(x)|^2)^{1/2}$. A convolver of $l^2$ is a function $g$ on $G$ such that for each $f \in l^2$ the convolution

$$(g * f)(x) = \sum_{y \in G} g(xy^{-1})f(y)$$

is defined and belongs to $l^2(G)$.

In accordance with the terminology of Eymard [3], we shall denote the space of "convolvers" by $VN(G)$. The norm of an element of $VN(G)$ will be the norm of the corresponding convolution operator (which is necessarily continuous) on $l^2(G)$. It is clear that $VN(G) \subseteq l^2(G)$. In this paper we study subsets $E \subseteq G$ with the property that every function $g \in l^2(G)$ supported on $E$ is a convolver. The existence of infinite sets $E$ satisfying this property was first established by M. Leinert [7]. He proved that if a set $E$ satisfies a certain condition, which we shall call Leinert's condition, then every square summable function $f$ supported on $E$ is a convolver, and

$$\|f\|_{VN(G)} \leq 5 \|f\|_{l^2(G)}.$$

The purpose of this paper is to give an alternate proof of Leinert's theorem which improves the constant $\sqrt{5}$. We prove that if $E$ satisfies Leinert's condition, and $f$ is supported on $E$, then

$$\|f\|_{VN(G)} \leq 2\|f\|_{l^2(G)}.$$

We also show that the constant 2 is the best possible if $E$ is an infinite set. To prove our result we use estimates involving $L^p$-convolution norm in

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the sense which was considered in [10] and [1].

We remark that sets satisfying Leinert’s condition are always subsets of a free group with at least two generators. On the other hand every set with no relation among its members satisfies this condition. (See [7], [8].) These sets have been used in [5] to construct multipliers of $A(G)$ which are not elements of $B(G)$.

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2. For a finitely supported function $f$ defined on $G$ we set

$$
||f||_2^2 = (f * f^*)(1) = \text{tr}(f * f)^s
$$

for $s = 1, 2, \ldots$, where $(f * f^*)^s$ denotes the convolution power. It is not difficult to see that $||f||_2^s$ is a norm. From a theorem of I. Kaplansky ([6, Theorem 1.8.1], [2]) we also have $\lim_{s \to \infty} ||f||_2^s = ||f||_{VN(G)}$.

Definition 1. Let $E$ be a subset of $G$ and $\gamma$ a positive integer. We say that $E$ is of type $L_2\gamma$ if for every finite sequence $\{x_1, x_2, \ldots, x_{2\gamma}\}$ the following relation holds:

$$
\left(\sum_{i=1}^{2\gamma-1} x_i x_{i+1} \right)^\gamma = 1
$$

if $x_{i+j} \neq x_{i+j+1}$ for $j = 1, 2, \ldots, 2\gamma - 1$.

Definition 2. A set $E$ is said to satisfy Leinert’s condition if $E$ is of type $L_2\gamma$ for every natural $\gamma$.

We can now state our main results:

Theorem. (i) If $E$ is of type $L_2\gamma$ in a discrete group $G$, then $E$ is $\Lambda(2\gamma)$, i.e., $||f||_2 \leq C_{2\gamma}||f||_2$ for every function $f$ with support in $E$, where $C_{2\gamma}^{2\gamma} = (n + 1)^{-1/2\gamma}$.

(ii) If $E$ is an infinite set of type $L_2\gamma$, then

$$
\sup \{||f||_2 : \text{supp } f \subseteq E, ||f||_2 = 1\} = C_{2\gamma}
$$

and $C_{2\gamma}$ is the minimal constant for all infinite $\Lambda(2\gamma)$ sets.

Corollary. If $E \subseteq G$ is a set which satisfies Leinert’s condition, then $||f||_{VN(G)} \leq 2||f||_2$ and 2 is the minimal constant for all infinite $\Lambda(\infty)$ sets.

Proof of (i). Let $f$ be a function of the form

$$
f = \sum_{i=1}^N a_i \delta_{x_i}, \quad x_i \in E, ||f||_2 = 1.
$$

For a subset $A \subseteq \Phi^*$, $\Phi = \{1, 2, \ldots, N\}$ we define
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$p_n(A) = \text{tr} \sum_{i \in A; i = (i_1, i_2, \ldots, i_{2n})} a_{i_1} \overline{a}_{i_2} \cdots a_{i_{2n-1}} \overline{a}_{i_{2n}} \delta x_{i_1 x_{i_2}} \cdots \delta x_{i_{2n-1} x_{i_{2n}}} \delta^{-1}.$

Because the set $E$ is of type $L_{2n}$ we note that $p_n$ is a positive measure on subsets of $\Phi^{2n} = \Phi \cdot \ldots \cdot \Phi$. It follows by induction from the following facts:

1. $p_n(\{i\}) = 0$ if for $i = (i_1, i_2, \ldots, i_{2n}) \in \Phi^{2n}$, $i_k \neq i_{k+1}$ for $k = 1, 2, \ldots, 2n - 1$, and

2. $p_n(\{i\}) = |a_{i_{k_0}}|^2 p_{n-1}(\{i'\})$ if for some $1 \leq k_0 < 2n$, $i_{k_0} = i_{k_0+1}$ and $i' = (i_1, \ldots, i_{k_0-1}, i_{k_0+2}, \ldots, i_{2n-1}, i_{2n})$. Let

$$S^n = \|f\|_{2n}^2 = p_n(\Phi^{2n}),$$

$$A_k = \{i: i = (i_1, i_2, \ldots, i_{2n}), i_k = i_{k+1}\}$$

and

$$S^n_k = p_n(A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c),$$

where $A_m^c = \Phi^{2n} \setminus A_m$. Since $p_n(A_k) = S^n_{k-1}$ for every natural $k < 2n$, so we obtain

$$S^n = S^{n-1} + S^n_k.$$  \hspace{1cm} (3)

Since $p_n(A_1^c) = p_n(A_1^c \cap A_2) + p_n(A_1^c \cap A_2^c)$, but $p_n(A_1^c \cap A_2) \leq p_n(A_2) = S^n_{k-1}$, therefore

$$S^n_2 \leq S^{n-1} + S^n_3.$$  \hspace{1cm} (4)

Now because $p_n(A_1^c \cap A_2^c) = p_n(A_1^c \cap A_2^c \cap A_3) + S^n_4$ and $p_n(A_1^c \cap A_3) = p_{n-1}(A_1^c)$, we obtain

$$S^n_3 \leq S^n_4 + S^{n-1}.$$  \hspace{1cm} (5)

By that same argument we have

$$S^n_{k+1} \leq S^n_{k+2} + S^n_{k-1}.$$  \hspace{1cm} (6)

But the set $E$ is of type $L_{2n}$ so from (6) we obtain

$$S^n_{2n} = S^n_k = 0 \text{ for } n < k \leq 2n.$$  \hspace{1cm} (7)

Applying (6) and (7) we have
From (6) we obtain
\[ S_{n-1}^n \leq S_n + S_{n-2}^{n-1}. \] (9)

Since \( S^1 = 1, S^2 = 2 \) and \( S_2^3 \leq 3 \), therefore from (9) we have
\[ S_{n-1}^n \leq \binom{n}{1} \quad \text{for} \quad n > 2. \] (10)

By this same way, from (6) we obtain
\[ S_{n-2}^n \leq S_{n-1}^n + S_{n-3}^{n-1}. \] (11)

and from (4) and (10) and also \( S_2^4 \leq 9 \) we have
\[ S_{n-2}^n \leq \binom{n+1}{2} - \binom{n+1}{0} \quad \text{for} \quad n > 3. \] (12)

And now by the induction argument we obtain
\[ S_{n-k}^n \leq \binom{n+k-1}{k} - \binom{n+k-1}{k-2} \quad \text{for} \quad k \geq 2. \] (13)

Since the following equality is true:
\[ \frac{1}{n} \binom{2n-2}{n-1} + \binom{2n-3}{n-2} - \binom{2n-3}{n-4} = \frac{1}{n+1} \binom{2n}{n} \] (14)
we obtain from (13) and (3), by induction,
\[ S^n \leq \frac{1}{n+1} \binom{2n}{n} \] (15)

**Proof of (ii).** Let \( E \) be an infinite set of type \( L_{2n}; \ E = \{x_1, x_2, \ldots, \} \) and \( f_N = N^{-1/2} \sum_{i=1}^{N} \delta_{x_i}. \) We prove by induction that
\[ S^n(f_N) = \|f_N\|_{2n}^{2n} = C_{2n}^{2n} + R_n(N), \] (16)
where \( \lim_{N \to \infty} R_n(N) = 0. \) That fact follows from the formula
\[ S^n = p_n\left( \bigcup_{k=1}^{n} A_k \right) = \sum_{k=1}^{n} p_n(A_k) - \sum_{i_1 < i_2} p_n(A_{i_1} \cap A_{i_2}) + \cdots + (-1)^{n-1} p_n(A_1 \cap A_2 \cap \cdots \cap A_n). \] (17)

Note also that if \( i_m + 1 \neq i_{m+1} \) for \( m = 1, 2, \ldots, k-1, \) then
\[ p_n\left( \bigcap_{i=1}^{k} A_i \right) = S^{n-k}. \] (18)
and
\[(19)\]
\[p_n \left( \bigcap_{m=1}^{n} A_m \right) \to 0 \quad (N \to \infty),\]
if for some \( m < n, \ i_m + 1 = i_{m+1}. \) In order to prove (19), it suffices to note that
\[(20)\]
\[p_n (A_1 \cap A_2) \to 0 \quad (N \to \infty),\]
but
\[(21)\]
\[p_n (A_1 \cap A_2) = N^{-1} \left\| f_N \right\|_{2^{n-2}}^{2^2n-2} \to 0 \quad (N \to \infty).\]

We shall prove the induction step in (16) if we show that
\[(22)\]
\[K = \sum_{k \in \mathbb{Z}} (-1)^k D_k B_{n-k}^m\]
equals zero, where \( D_k = C_{2k}^{2k} \) and \( B_{n-k}^m \) denote the number of subsequences of the sequence \((1, 2, \ldots, n)\) of the form \((k_1, k_2, \ldots, k_m)\) where for every \(1 \leq s < m - 1, k_s + 1 \neq k_{s+1}.\) It is easy to see that
\[(23)\]
\[B_n^m = \binom{n + 1 - m}{m}.\]

Applying the following formulas (see [4])
\[(24)\]
\[\sum_{v=0}^{n} (-1)^v \binom{a}{v} = (-1)^n \binom{a - 1}{n},\]
\[(25)\]
\[\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{k} \binom{2k + 1}{v} = 0,\]
we obtain
\[K = \sum_{k \in \mathbb{Z}} (-1)^k \frac{1}{k + 1} \binom{2k}{k} \binom{k + 1}{n - k} = \frac{1}{n} \sum_{k} (-1)^k \binom{n}{k} \binom{2k}{n - 1} \]
\[= \frac{1}{n} \sum_{k} \sum_{v} (-1)^{n+k+v-1} \binom{n}{k} \binom{2k + 1}{v} = 0.\]

The Corollary follows at once from the following inequality (see [11]):
\[\frac{2^{2n-1}}{n} \leq \binom{2n}{n} \leq 2^{2n-1}.

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