ON PAIRS OF NONINTERSECTING FACES OF CELL COMPLEXES

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ABSTRACT. We show that, for all cell complexes whose underlying set is a manifold, \( M \), an alternating sum of numbers of pairs of faces that do not intersect is a topological invariant. This is done by proving that it is a function of the Euler characteristic, \( \chi \), of \( M \).

A cell complex [1, pp. 39–40] is a finite family, \( \mathcal{C} \), of polytopes in \( \mathbb{R}^n \) such that

(i) every face of a member of \( \mathcal{C} \) is itself a member of \( \mathcal{C} \);
(ii) the intersection of any two members of \( \mathcal{C} \) is a face of each of them.

We shall call a polytope \( P \in \mathcal{C} \) a face of \( \mathcal{C} \). The number of \( i \)-dimensional faces of \( \mathcal{C} \) will be denoted by \( f_i \). The subset of \( \mathbb{R}^n \) consisting of all the points of members of \( \mathcal{C} \) will be denoted by set \( \mathcal{C} \). The boundary complex of a \((d + 1)\)-dimensional polytope, \( P \), is the set of all faces of \( P \) of at most dimension \( d \).

Let \( \mathcal{C} \) be any cell complex such that set \( \mathcal{C} = M \) where \( M \) is a \( d \)-dimensional manifold. Then \( \mathcal{C} \) will obey Euler's relation

\[
\chi(M) = \sum_{i=0}^{d} (-1)^i f_i.
\]

If \( \mathcal{C} \) is the boundary complex of a \((d + 1)\)-polytope then \( M \) will be homeomorphic to the surface of a hypersphere and

\[
\chi(M) = 1 + (-1)^d.
\]

Let \( \alpha_{ij} \) = the number of ordered pairs of faces of \( \mathcal{C} \) of dimensions \( i \) and \( j \) that do not intersect. Then the \( \alpha_{ij} \) are called incidence numbers of \( \mathcal{C} \). Note that \( \alpha_{ij} = \alpha_{ji} \). We are interested in the sum

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\(\psi(C) = \sum_{i=0}^{d} \sum_{j=0}^{d} (-1)^{i+j}\alpha_{ij}\)

This and similar quantities have been investigated by Wu [2] and others.

Now we assign to each of the \(f_i\) \(i\)-dimensional faces of \(C\) a different number \(k (k = 1, 2, 3, \ldots, f_i)\). Then let \(p_{ij}(k)\) = the number of \(j\)-dimensional faces of \(C\) intersecting with the \(i\)-dimensional face of \(C\) assigned the number \(k\). Then

\[\alpha_{ij} = \sum_{k=1}^{f_i} (f_j - p_{ij}(k)) = f_j - \sum_{k=1}^{f_i} p_{ij}(k).\]

We shall now prove

**Theorem I.** If \(R\) is the \(i\)-dimensional face of \(C\) assigned the number \(k (0 \leq i \leq d, \ 1 \leq k \leq f_i)\) then

\[\sum_{j=0}^{d} (-1)^{j} p_{ij}(k) = (-1)^d.\]

**Proof.** Construct around \(R\) a figure \(Q'\) homeomorphic to the surface of a \(d\)-dimensional hypersphere and which contains exactly all the vertices of \(R\) within its interior. Then the intersection of the faces of \(C\) with \(Q'\) define a topological polytope, \(Q\), of dimension \(d\).

Let

\[r_j = \begin{cases} 
\# \text{ of } j\text{-dimensional faces of } R & \text{if } j < i, \\
1 & \text{if } j = i, \\
0 & \text{if } j > i,
\end{cases}\]

\[q_j = \begin{cases} 
\# \text{ of } j\text{-dimensional faces of } Q & \text{if } 0 \leq j \leq d - 1, \\
0 & \text{if } j = -1.
\end{cases}\]

Then, since every \(j\)-dimensional face of \(C\) emanating from (intersecting but not contained in) \(R\) intersects \(Q'\) in a \((j - 1)\)-dimensional face of \(Q\), it is not hard to see that for all \(0 \leq j \leq d\),

\[p_{ij}(k) = q_{j-1} + r_j.\]

Since \(Q\) is a \(d\)-dimensional topological polytope, and \(R\) is an \(i\)-dimensional one, by (2) we have
\[
\sum_{j=0}^{d-1} (-1)^j q_j = 1 + (-1)^{d-1} = 1 - (-1)^d,
\]

Then, by (6) and (7), (5) is true. Q.E.D.

**Theorem II.** \( \psi(\overline{C}) = \chi^2(M) - \chi(M). \)

**Proof.** From (3) and (4),

\[
\psi(\overline{C}) = \sum_{i=0}^{d} \sum_{j=0}^{d} (-1)^{i+j} f_{ij} - \sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{k=1}^{d} (-1)^{i+j+p_{ij}(k)}.
\]

From (1) we get

\[
\chi^2(M) = \left( \sum_{i=0}^{d} (-1)^i f_i \right)^2 = \sum_{i=0}^{d} \sum_{j=0}^{d} (-1)^{i+j} f_{ij}.
\]

And by (5) and (1)

\[
\sum_{i=0}^{d} \sum_{j=0}^{d} \sum_{k=1}^{d} (-1)^{i+j+p_{ij}(k)} = \sum_{i=0}^{d} (-1)^i \sum_{k=1}^{d} \sum_{j=0}^{d} (-1)^j p_{ij}(k) = (-1)^d \chi(M),
\]

so that (8) becomes \( \psi(\overline{C}) = \chi^2(M) - (-1)^d \chi(M) \). And, since \( \chi(M) = 0 \) whenever \( d \) is odd \( \psi(\overline{C}) = \chi^2(M) - \chi(M) \). Q.E.D.

**Corollary.** If \( \overline{C} \) is the boundary complex of a \((d+1)\)-dimensional polytope, \( \psi(\overline{C}) = 1 + (-1)^d \).

**REFERENCES**


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