NONFACTORIZATION OF FUNCTIONS IN BANACH SUBSPACES OF $L^1(G)$

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ABSTRACT. In this note we first prove a theorem on factorization of functions in certain subsets of $L^1(G)$, where $G$ is a nondiscrete locally compact Abelian group with dual group $\hat{G}$. One of the corollaries of this theorem answers a question of R. Larsen concerning the algebras of functions with Fourier transforms in $L^p(\hat{G})$. The other corollaries contain nonfactorization results which sharpen some known theorems.

Throughout this note $G$ denotes a nondiscrete locally compact Abelian group with dual group $\hat{G}$. For $A, B \subset L^1(G)$, the sets $\{f \ast g: f \in A, g \in B\}$ and $\{\sum_{i=1}^{n} f_i \ast g_i: f_i \in A, g_i \in B, n = 1, 2, 3, \cdots\}$ will be denoted by $A \ast B$ and $[A \ast B]$, respectively. For $1 \leq p < \infty$, define $A^p(G) = \{f \in L^1(G): \hat{f} \in L^p(\hat{G})\}$ and $\|f\|_{A^p} = \|f\|_1 + \|\hat{f}\|_p$ for $f \in A^p(G)$. The Banach algebras $(A^p(G), \|\cdot\|_{A^p})$ have been studied by many authors (see the survey article by Larsen [2]). For $1 \leq p < q < \infty$, we have $A^q(G) \ast A^p(G) \subset A^p(G)$, and Larsen [2] has raised the question: When, if ever, is $A^q(G) \ast A^p(G) = A^p(G)$? The answer to this question is given in a corollary (see Corollary 1 below) of the following theorem, the proof of which is based on an idea first used in Martin and Yap [3, p. 218].

Theorem 1. Let $A, B$ be subsets of $L^1(G)$ such that $\hat{A} = \{\hat{f}: f \in A\} \subset L^p(\hat{G})$ for some $p \in (0, \infty)$ and $B \subset [A \ast B]$. Then $\hat{B} \subset L^r(\hat{G})$ for all $r \in (0, \infty)$.

Proof. It is clear that $\hat{B} \subset L^r(\hat{G})$ for all $r \in [p, \infty]$. Now consider $h \in B$ and $r \in (0, p)$. Choose a positive integer $N$ such that $p/2^N \leq r$. By repeated use of the condition $B \subset [A \ast B]$, we can write $h = h_1 + \cdots + h_m$ with $h_i = f_{i_1} \ast f_{i_2} \ast \cdots \ast f_{2^N} \ast g$, where $f_{i} \in A$, $g \in B$. Since $\hat{f}_{i} \in L^p(\hat{G})$, it follows from Hölder's inequality that

$$\left(\sum_{i=1}^{m} f_{i_1} \ast f_{i_2} \ast \cdots \ast f_{2^N}\right)^r \leq \sum_{i=1}^{m} \left(\sum_{n=1}^{\infty} |\hat{f}_{i_n}|^r\right)^{r/p} \leq \sum_{i=1}^{m} \left(\sum_{n=1}^{\infty} \|\hat{f}_{i_n}\|_p^r\right)^{r/p}.$$
is in $L^{p/2N} (\hat{G})$. Since $\hat{g}$ is bounded, we have $\hat{h}_i \in L^{p/2N} (\hat{G})$ and hence $\hat{h}_i \in L^r (\hat{G})$. Thus $\hat{h} \in L^r (\hat{G})$ for all $r \in (0, p)$. This completes the proof.

We now give some sample corollaries of Theorem 1.

Corollary 1. Let $1 \leq p < q < \infty$. Then $[A^q(G) * A^p(G)]$ is a proper subspace of $A^p(G)$.

Proof. Since $A^p(G)$ is a "character" Segal algebra, it follows that $A^p(G)^* \not\subset L^r (\hat{G})$ for some $r$ in $(0, \infty)$ (see Wang [4] for details). The conclusion, in view of Theorem 1, is now clear.

For $1 < p \leq \infty$, define $B^p(G) = L^1 (G) \cap L^p (G)$ and $\|f\|_{B^p} = \|f\|_1 + \|f\|_p$ for $f \in B^p(G)$. It is well known [5], [4], [1] that $[B^p(G) * B^q(G)]$ is a proper subspace of $B^p(G)$ for $1 < p < \infty$ (and $G$ nondiscrete). Now we can prove the following stronger result.

Corollary 2. For $1 < p < q < \infty$, $[B^p(G) * B^q(G)]$ is a proper subspace of $B^q(G)$.

Proof. Clearly $[B^p(G) * B^q(G)] \subset B^q(G)$. By the Hausdorff-Young theorem we have $B^p(G)^* \subset L^r (\hat{G})$ for some $r$. The desired conclusion follows immediately from Theorem 1 and the fact that $B^q(G)$ is a "character" Segal algebra (see Wang [4]).

Let $T$ be the circle group. For each positive integer $k$, define $C^k(T)$ to be the space of all functions with $k$ continuous derivatives, and norm $C^k(T)$ by

$$\|f\|_{C^k} = \max_{0 \leq j \leq k} \max_{x \in T} |f^{(j)}(x)|.$$ 

It is easy to see that we have the chain [4, p. 234]

(1) $\cdots \subset C^{k+1}(T) \subset C^k(T) \subset \cdots \subset C(T) \subset \cdots \subset L^r(T) \subset L^s(T) \subset \cdots$ ,

where $r > s > 1$. Even though $C^k(T)$, $k \geq 1$, is not a "character" Segal algebra, we nevertheless have $C^k(T)^* \not\subset L^r(\hat{T})$ for some $r$ in $(0, \infty)$ (see [4]). The following corollary, in view of Theorem 1, is now immediate.

Corollary 3. If $A$, $B$ are any two sets from the chain (1) with $A \supset B$, then $[A * B]$ is a proper subspace of $B$.

Remark. The special case $A = B$ of Corollary 3 was obtained by Wang [4]. It is now clear that all the nonfactorization results in Wang's paper can be extended in the same way.

From Theorem 1 we see that if $A$, $B$ are subsets of $L^1(G)$ with
\[ \hat{A} \subset L^p(\hat{G}), \hat{B} \not\subset L^r(\hat{G}) \] for some \( p, r \) in \((0, \infty)\), then \( B \not\subset [A \ast B] \). Thus, for the purpose of obtaining nonfactorization results, it is of interest to have conditions on \( B \) which would imply \( \hat{B} \not\subset L^r(\hat{G}) \) for some \( r \) in \((0, \infty)\). The condition (called Property P) given by Wang [4, p. 235] for \( L^1 \)-dense Banach subalgebras \((B, \| \cdot \|_B)\) of \( L^1(G) \) is also meaningful when \((B, \| \cdot \|_B)\) is a Banach subspace of \( L^1(G) \). The following extension of the nonfactorization theorem in Wang [4, Theorem 4.1] can be proved by using Theorem 1 above and the idea used in Wang’s proof.

**Theorem 2.** Let \( A \) be a subset of \( L^1(G) \) with \( \hat{A} \subset L^p(\hat{G}) \) for some \( p \) in \((0, \infty)\). Let \( B \) be a subspace of \( L^1(G) \) such that \((B, \| \cdot \|_B)\) is Banach space having Property P and \( \| \cdot \|_1 \leq M \| \cdot \|_B \) for some constant \( M \). Then \( B \not\subset [A \ast B] \).

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**REFERENCES**