APPROXIMATING ZEROS OF ACCRETIVE OPERATORS

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ABSTRACT. Let $A$ be an $m$-accretive set in a reflexive Banach space $E$ with a Gateaux differentiable norm. For positive $r$ let $J_r$ denote the resolvent of $A$. If the duality mapping of $E$ is weakly sequentially continuous and $0$ is in the range of $A$, then for each $x$ in $E$ the strong $\lim_{r \to \infty} J_r x$ exists and belongs to $A^{-1}(0)$. This is an extension to a Banach space setting of a result previously known only for Hilbert space.

Let $H$ be a real Hilbert space and $U \subseteq H \times H$ a maximal monotone operator. For each positive $r$ there is a unique $y_r$ in $H$ such that $0 \in y_r + rU(y_r)$. It is known [4] that if $0$ belongs to the range of $U$, then the strong $\lim_{r \to \infty} y_r$ exists and is the point of $U^{-1}(0)$ closest to $0$. It is our purpose in this note to extend this result to accretive operators in certain Banach spaces. According to [4], this leads to the possibility of calculating a zero of the given operator as the limit of an iteratively constructed sequence. Our method of proof is not a direct generalization of the Hilbert space proof. It works, however, only in a restricted class of Banach spaces. The question of whether our theorem is valid in other Banach spaces remains open.

Let $E^*$ denote the dual of a real Banach space $E$. The duality mapping $J$ from $E$ into the family of nonempty subsets of $E^*$ is defined by

$$J(x) = \{x^* \in E^*: (x, x^*) = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}.$$ 

$J$ is single-valued if and only if the norm of $E$ is Gateaux differentiable. If $A$ is a subset of $E \times E$ and $x \in E$, we define

$$Ax = \{y \in E: [x, y] \in A\}$$

and set

$$D(A) = \{x \in E: Ax \neq \emptyset\}.$$ 

The range of $A$ is defined by

$$R(A) = \bigcup\{Ax: x \in D(A)\}.$$ 

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and its inverse by

\[ A^{-1}y = \{x \in E : y \in Ax \}. \]

I will stand for the identity operator on \( E \). The closure of a subset \( D \) of \( E \) will be denoted by \( \text{cl}(D) \). A mapping \( T : D \to E \) is said to be nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x \) and \( y \) in \( D \). In the sequel, \( \to \) and \( \rightharpoonup \) will denote strong and weak convergence respectively.

A subset \( A \) of \( E \times E \) is called accretive [7] if for all \( x_i \in D(A) \) and \( y_i \in Ax_i, \ i = 1, 2, \) there exists \( j \in J(x_1 - x_2) \) such that \( \langle y_1 - y_2, j \rangle \geq 0 \).

Let \( D \) be a subset of \( E \) and \( A \) an accretive set (= accretive operator) with \( D(A) \subseteq D \). \( A \) is said to be maximal accretive in \( D \) if there is no proper accretive extension \( B \) of \( A \) with \( D(B) \subseteq D \). An accretive set \( A \) is maximal accretive if it is maximal accretive in \( E \). It is \( m \)-accretive if \( R(I + A) = E \). (It follows that \( R(I + rA) = E \) for all positive \( r \).) If \( T : E \to E \) is nonexpansive, then \( I - T \) is \( m \)-accretive. If \( A \) is \( m \)-accretive, then it is maximal accretive, but the converse is not true in general. If \( A \) is accretive one can define, for each \( r > 0 \), a nonexpansive single-valued mapping \( J_r : R(I + rA) \to D(A) \) by \( J_r = (I + rA)^{-1} \). It is called the resolvent of \( A \). Conditions which imply that an \( m \)-accretive set is surjective can be found in [10].

The duality mapping \( J \) of a Banach space \( E \) with a Gâteaux differentiable norm [5] is said to be weakly sequentially continuous if \( x_n \to x \) in \( E \) implies that \( \{J(x_n)\} \) converges weak star to \( J(x) \) in \( E^* \). This happens, for example, if \( E \) is a Hilbert space, or finite-dimensional and smooth, or \( l_p, 1 < p < \infty \). This property of Banach spaces was introduced by Browder [1]. More information can be found in [6].

**Lemma.** Let \( A \) be a maximal accretive set in a Banach space \( E \) whose norm is Gâteaux differentiable. Let \( x_n \in D(A) \), \( y_n \in Ax_n \), \( x_n \to x \), and \( y_n \to y \). If the duality mapping \( J \) is weakly sequentially continuous, then \([x, y] \in A\).

**Proof.** Let \( z \in D(A) \) and \( w \in Az \). We have

\[
|\langle y_n - w, J(x_n - z) \rangle - (y - w, J(x - z))| \\
\leq |\langle y_n - y, J(x_n - z) \rangle| + |\langle y - w, J(x_n - z) - J(x - z) \rangle| \\
\leq \|y_n - y\| \|x_n - z\| + |\langle y - w, J(x_n - z) - J(x - z) \rangle|.
\]

Thus

\[
(y - w, J(x - z)) = \lim_{n \to \infty} (y_n - w, J(x_n - z)) \geq 0.
\]

The result follows.
A closed subset $C$ of a Banach space $E$ is called a nonexpansive retract of $E$ if there exists a retraction of $E$ onto $C$ which is a nonexpansive mapping. A retraction $P: E \to C$ is called a sunny retraction if $P(x) = v$ implies that $P(v + r(x - v)) = v$ for all $x \in E$ and $r \geq 0$. (We prefer this term to the one used by Brück [3] because suns already occur in approximation theory.) If there exists a retraction $P: E \to C$ which is both sunny and nonexpansive, then $C$ is said to be a sunny nonexpansive retract of $E$. If $C$ is a sunny nonexpansive retract of a Banach space whose norm is Gâteaux differentiable, then the sunny nonexpansive retraction on $C$ is unique [3, Theorem 1], [8, Lemma 2.7]. The metric projection on a closed and convex subset of a Hilbert space is both sunny and nonexpansive.

**Theorem.** Let $A$ be an $m$-accretive set in a reflexive Banach space $E$ with a Gâteaux differentiable norm. If the duality mapping $J$ of $E$ is weakly sequentially continuous and $0 \in R(A)$, then for each $x$ in $E$ the strong $\lim_{r \to \infty} J_{\frac{x}{r}}$ exists and belongs to $A^{-1}(0)$.

**Proof.** Let the positive sequence $\{r_n : n = 1, 2, \ldots \}$ tend to infinity. Let $x \in E$ and $y \in A^{-1}(0)$. Set $x_n = J_{r_n} x$. We have $(x_n - x, J(y - x_n)) \geq 0$ because $(x - x_n)/r_n$ belongs to $Ax_n$ and $0 \in Ay$. Consequently,

$$\|y - x_n\|^2 \leq (y - x, J(y - x_n)) \leq \|y - x\| \|y - x_n\|$$

and $\{x_n\}$ is bounded. Let $Px$ be the weak limit of a subsequence $\{x_k\}$ of $\{x_n\}$. Clearly $(x - x_k)/r_k \to 0$. By the Lemma, $[Px, 0]$ belongs to $A$. Therefore

$$\|Px - x_k\|^2 \leq (Px - x, J(Px - x_k)) \to 0.$$ 

Thus $\{x_k\}$ converges strongly to $Px$. It follows that $(Px - x, J(y - Px)) \geq 0$ for all $x$ in $E$ and $y$ in $A^{-1}(0)$. In other words [8, Lemma 2.7], $P: E \to A^{-1}(0)$ is both sunny and nonexpansive. Since $P$ is necessarily unique, the sequence $\{x_n\}$ itself converges strongly to $Px$. This completes the proof.

**Corollary (cf. [8, Theorem 3.2]).** Let $T$ be a nonexpansive self-mapping of $E$, a reflexive Banach space with a Gâteaux differentiable norm. Suppose that $T$ has a nonempty fixed point set and that $E$ has a weakly sequentially continuous duality mapping. Let $x$ belong to $E$. For each $0 < k < 1$ let $x_k$ satisfy $x_k = kTx_k + (1 - k)x$. Then the strong $\lim_{k \to 1} x_k$ exists and is a fixed point of $T$.

In the course of the proof of the Theorem it has been established that $A^{-1}(0)$ is a nonexpansive retract of $E$. Since $A^{-1}(0)$ is the fixed point set
of the nonexpansive mapping \( J_r \) (for all \( r > 0 \)), this is also a consequence of [2, Theorem 2]. In a similar setting, \( c_1(D(A)) \) is also a nonexpansive retract of \( E \) [9, Theorem 3.7].

Remark. A version of the Theorem is true for a certain class of accretive operators which are not necessarily \( m \)-accretive.

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REFERENCES