A NOTE ON DILWORTH'S EMBEDDING THEOREM

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ABSTRACT. The dimension of a poset $X$ is the smallest positive integer $t$ for which there exists an embedding of $X$ in the cartesian product of $t$ chains. R. P. Dilworth proved that the dimension of a distributive lattice $L = 2^X$ is the width of $X$. In this paper we derive an analogous result for embedding distributive lattices in the cartesian product of chains of bounded length. We prove that for each $k \geq 2$, the smallest positive integer $t$ for which the distributive lattice $L = 2^X$ can be embedded in the cartesian product of $t$ chains each of length $k$ equals the smallest positive integer $t$ for which there exists a partition $X = C_1 \cup C_2 \cup \cdots \cup C_t$ where each $C_i$ is a chain of at most $k - 1$ points.

1. Preliminaries. A poset consists of a pair $(X, P)$ where $X$ is a set and $P$ is a reflexive, antisymmetric, and transitive relation on $X$. The notations $(x, y) \in P$ and $x \leq y$ in $P$ are used interchangeably. If $x$ and $y$ are distinct points in $X$ and neither $(x, y)$ nor $(y, x)$ is in $P$, then we say $x$ and $y$ are incomparable and write $x \not< y$. For convenience we will frequently use a single symbol to denote a poset. If $X$ and $Y$ are isomorphic posets, then we write $X = Y$ and if $X$ is isomorphic to a subposet of $Y$, then we write $X \subseteq Y$. The dual of a poset $X$, denoted $X^\circ$, is the poset on the same set with $x \leq y$ in $X^\circ$ iff $y \leq x$ in $X$.

If $(X, P)$ and $(Y, Q)$ are posets, their free sum, denoted $X + Y$, is the poset $(X \cup Y, P \cup Q)$ where $\cup$ denotes disjoint union. Their cartesian product $X \times Y$ is the poset $(X \times Y, S)$ where $S = \{(x, y), (z, w)\}: x \leq z$ in $X$ and $y \leq w$ in $Y$. The cartesian product of $n$ copies of $X$ is denoted $X^n$.

The join of $(X, P)$ and $(Y, Q)$, denoted $X \oplus Y$, is the poset $(X \cup Y, P \cup Q \cup X \times Y)$. A function $f: Y \to X$ is order preserving iff $y \leq w$ in $Y$ implies $f(y) \leq f(w)$ in $X$. The cardinal power of $X$ and $Y$, denoted $X^Y$, is the poset consisting of all ordering preserving functions from $Y$ to $X$ with $f \leq g$ in $X^Y$ iff $f(y) \leq g(y)$ in $X$ for every $y \in Y$.

A poset $C$ for which $x, y \in C$ imply $x \leq y$ or $y \leq x$ is called a chain. We denote the $n$ element chain $0 < 1 < 2 < \cdots < n - 1$ by $n$. A chain $(X, L)$ is said to be linear extension of $(X, P)$ when $P \subseteq L$. We also say $L$ is a linear extension of $P$. By a theorem of Szpilrajn [12], if $C$ denotes the collection of all linear extensions of $P$, then $\bigcap C = P$. 

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A poset $A$ for which $x, y \in A$ and $x \neq y$ imply $x \nprec y$ is called an antichain. We denote an element antichain by $\bar{n}$. The width of a poset $X$, denoted $W(X)$, is the number of elements in a maximum antichain in $X$.

The justification for the exponential notation for the cardinal power of posets is given by the following property (see [2] for details).

**Fact 1.** $X^{Y \times Z} = X^Y \times X^Z$.

In this paper we are concerned primarily with cardinal powers of the form $2^X$. For such posets, we have

**Fact 2.** $2^{\bar{n}} = n + 1$ and $2^{\bar{n}} = 2^n$.

If $(X, P)$ and $(Y, Q)$ are posets, $X = Y$, and $P \subseteq Q$, then it is easy to see that $2^Y \subseteq 2^X$. In fact a stronger result holds.

**Lemma 1.** Let $(X, P)$ and $(Y, Q)$ be posets, $Y \subseteq X$, and $P \cap (Y \times Y) \subseteq Q$. Then $2^Y \subseteq 2^X$.

**Proof.** Define a function $F: 2^Y \to 2^X$ by $F(f)(x) = f(x)$ if $x \in Y$, $F(f)(x) = 0$ if $x \in X - Y$ and there exists $y \in Y$ such that $y \npreceq x$ in $X$ and $f(y) = 0$, and $F(f)(x) = 1$ otherwise. It is straightforward to verify that $F$ is an embedding.

2. **Introduction.** Dushnik and Miller [5] defined the dimension of a poset $X$, denoted $\text{Dim } X$, as the smallest positive integer $t$ for which there exist $t$ linear extensions $L_1, L_2, \ldots, L_t$ of the partial ordering $P$ on $X$ such that $L_1 \cap L_2 \cap \cdots \cap L_t = P$. Ore [9] gave an equivalent definition of $\text{Dim } X$ as the smallest positive integer $t$ for which $X \subseteq C_1 \times C_2 \times \cdots \times C_t$, where each $C_i$ is a chain.

A very important example of a poset is a distributive lattice for which we have the following well-known representation theorem: A poset $M$ is a distributive lattice iff $M = 2^X$ for some poset $X$. In 1950, R. P. Dilworth [4] published the following theorem giving the dimension of a distributive lattice.

**Theorem 1.** $\text{Dim } 2^X = W(X)$.

In order to prove Theorem 1, Dilworth derived his famous decomposition theorem.

**Theorem 2.** If $X$ is a poset and $W(X) = n$, then the point set $X$ can be partitioned into $n$ subsets $C_1, C_2, \ldots, C_n$ such that the subposet determined by each $C_i$ is a chain.

Compact proofs of Theorem 2 appear in [10] and [15] and Theorem 1 is also discussed in [11].

In this paper we generalize the concept of dimension for posets to obtain an extension of Theorem 1. For an integer $k \geq 2$, we define the $k$-dimension of a poset $X$, denoted $\text{Dim}_k X$, as the smallest positive integer $t$ for which $X \subseteq k^t$. 
3. Some elementary inequalities. In [13], the inequality $\text{Dim}_2 X \leq |X|$ for all $X$ is established and the family of posets for which equality holds is determined. In [14], the inequalities $\text{Dim}_3 X \leq \frac{|X|}{2}$ for $|X| \geq 5$ and $\text{Dim}_4 X \leq \frac{|X|}{2}$ for $|X| \geq 6$ are established. Hiraguchi [6] proved that $\text{Dim} X \leq \frac{|X|}{2}$ for $|X| \geq 4$ and Bogart and Trotter [3] and Kimble [8] determined the collection of all posets for which equality holds.

Clearly $\text{Dim} X < \text{Dim}^e X$ and since $kl \leq k + l$, we have $\text{Dim}^e X < \text{Dim}^e X$. Since there are $k'$ points in $k'$, we have $\text{Dim}_k X \geq \log_k |X|$ and since the longest chain in $k'$ has length $(k - 1)t + 1$, we conclude $\text{Dim}_k X = (n - 1)/(k - 1)$. It is also easy to compute $\text{Dim}_m X$ by the methods compiled by Katona [6].

Theorem 3. $\text{Dim}_k X \leq 2 \text{Dim}_{k+1} X$.

Proof. Suppose $\text{Dim}_{k+1} X = t$ and let $f: X \to k+l$ be an embedding. Define $g: X \to k^{2t}$ by:

$$g(x)(i) = \begin{cases} 
  f(x)(i) - 1 & \text{when } f(x)(i) > 0 \text{ and } i \leq t, \\
  0 & \text{when } f(x)(i) = 0 \text{ and } i \leq t, \\
  f(x)(i) & \text{when } f(x)(i) < k \text{ and } i > t, \\
  k - 1 & \text{when } f(x)(i) = k \text{ and } i > t.
\end{cases}$$

It follows easily that $g$ is an embedding and thus $\text{Dim}_k X < 2t$.

In order to determine whether or not the inequality of Theorem 3 is best possible, we need the following generalization of a well-known property (see [2, problem 7, p. 101]) of dimension which we state without proof.

Fact 4. If $X$ and $Y$ are posets, then $\text{Dim}_k X \times Y \leq \text{Dim}_k X + \text{Dim}_k Y$. If $X$ and $Y$ have distinct greatest and least elements, then equality holds.

Since $\text{Dim}_k k+1 = 2$ and $\text{Dim}_{k+1} k+1 = 1$, it follows from Fact 4 that $\text{Dim}_k k+1 = 2t$ while $\text{Dim}_{k+1} k+1 = t$ for all $t \geq 1$.

4. Dilworth's embedding theorem. A short proof of Dilworth's embedding theorem (Theorem 1) is given here for the sake of completeness. We assume Theorem 2.

To show that $\text{Dim}_2 X \leq \text{W}(X)$, let $|X| = m$, $\text{W}(X) = n$, and $X = C_1 \cup C_2 \cup \ldots \cup C_n$ be a decomposition into chains. It follows that

$$2^X \subseteq 2^{C_1 + C_2 + \cdots + C_n} = 2^{C_1} \times 2^{C_2} \times \cdots \times 2^{C_n} \subseteq m + 1 \bigwedge$$

and thus $\text{Dim}_2 X \leq n$.

On the other hand, if $A$ is an antichain of $X$ with $|A| = n$, then $2^n = 2^A \subseteq 2^X$ and we conclude that $\text{Dim}_2 X \geq \text{Dim}_2 X = n$.

The reader is invited to compare this argument with the proof of Theorem 3 in [13].
5. Some additional inequalities. For a poset $X$ and an integer $m \geq 1$, let $P_m(X)$ be the smallest positive integer $t$ for which there exists a partition of the point set of $X$ of the form $X = C_1 \cup C_2 \cup \cdots \cup C_t$ where the subposet determined by each $C_i$ is a chain with $|C_i| \leq m$. The first half of the argument given in the preceding section allows us to conclude that $\text{Dim}_{k-1}^X \leq P_{k-1}(X)$.

Now every poset $Y$ can be written as the free sum $Y = Y_1 + Y_2 + \cdots + Y_r$ of its components. For a poset $Y$ with components $Y_1, Y_2, \ldots, Y_r$ and an integer $m \geq 1$, we then define $S_m(Y) = \sum_{i=1}^r |Y_i|/m!$. To provide a generalization of the concept of width, we define $W_m(X) = \max\{S_m(Y) : Y \subseteq X\}$. Dilworth's decomposition theorem can then be restated in the following form.

**Theorem 4.** For every poset $X$, there exists an integer $m_0$ such that $m \geq m_0$ implies $P_m(X) = W_m(X)$.

To see the connection between these definitions and Dilworth's embedding theorem we observe that the following result holds.

**Theorem 5.** For every poset $X$ and every integer $k \geq 2$, $W_{k-1}(X) \leq \text{Dim}_k^X \leq P_{k-1}(X)$.

**Proof.** Choose a subposet $Y \subseteq X$ with $W_{k-1}(X) = S_{k-1}(Y)$; let the components of $Y$ be $Y_1, Y_2, \ldots, Y_r$ and for each $i \leq r$ let $C_i$ be a linear extension of $Y_i$. If follows that

$$2^1 \times 2^1 \times \cdots \times 2^1 \subseteq 2^1 \times 2^2 \times \cdots \times 2^r = 2^{Y_1 + Y_2 + \cdots + Y_r} = 2^X$$

and therefore

$$\text{Dim}_k^X \left(2^1 \times 2^1 \times \cdots \times 2^1\right) \leq \text{Dim}_k^X.$$

$$\text{Dim}_k^X \left(2^1 \times 2^1 \times \cdots \times 2^1\right) = \sum_{i=1}^r |C_i|/(k-1) = \sum_{i=1}^r |Y_i|/(k-1) = S_{k-1}(Y) = W_{k-1}(X).$$

For $m = 1$, $W_1(X) = P_1(X) = |X|$ for all $X$. It is also true that $W_2(X) = P_2(X)$ for all $X$; in fact a more general result holds which we outline here. For a graph $H$ with components $H_1, H_2, \ldots, H_r$, let $S_m(H) = \sum_{i=1}^r |H_i|/m!$. For a graph $G$, let $W_m(G) = \max\{S_m(H) : H \text{ is an induced subgraph of } G\}$. Also let $P_m(G)$ be the smallest positive integer $n$ for which there exists a partition of the vertex set of $G$ into $n$ subsets so that the induced subgraph spanned by each subset is a complete graph on at most $m$ vertices.
For a poset $X$ the comparability graph of $X$, denoted $G_X$, is the graph whose vertex set is the point set of $X$ with distinct points $x$, $y \in X$ adjacent in $G_X$ iff $x < y$ or $y < x$ in $X$. Clearly $P_m(X) = P_m(G_X)$ and $W_m(X) = W_m(G_X)$.

**Theorem 6.** $W_2(G) = P_2(G)$ for all graphs.

**Proof.** We assume Hall's matching theorem for graphs and then proceed by induction on $|X|$. Now suppose $G$ is a graph with $W_2(G) = t$ and let $H$ be a subgraph of $G$ with components $H_1, H_2, \ldots, H_r$ so that $W_2(G) = W_2(H) = \sum_{i=1}^r |H_i|/2 = t$. We further assume that $H$ is chosen so that $r$ is maximal and $|H|$ is minimal. Thus $W_2(H_i - x) < W_2(H_i)$ for every $i \leq r$ and every $x \in H_i$ and we may assume that $H \neq X$.

Now construct a bipartite graph $(X, Y)$ with $X = \{v_1, v_2, \ldots, v_s\}$ and $Y = G - H$. A vertex $y \in Y$ is adjacent to $v_i$ in $(X, Y)$ iff $y$ is adjacent to at least one vertex of $H_i$ in $G$.

By Hall's matching theorem, there exists a matching of $Y$ into $X$ for if $Y_1 \subseteq Y$, $X_1 = \{v \in X : v \perp y$ for some $y \in Y_1\}$, and $|X_1| < |Y_1|$, then $W_2(H \cup Y_1) > W_2(H)$.

We then assume that the elements of $Y$ are labeled so that $Y = \{y_1, y_2, \ldots, y_s\}$, $s \leq r$, and $y_i \perp H_i$ in $(X, Y)$ for each $i \leq s$. We then choose vertices $a_1, a_2, \ldots, a_s$ from $H_1, H_2, \ldots, H_s$ so that $y_i \perp a_i$ in $G$ for each $i \leq s$. From the inductive hypothesis, we conclude that for each $1 \leq i \leq s$, the subgraph $H_i \setminus a_i$ can be partitioned into $W_2(H_i) - 1$ complete subgraphs each of at least two vertices.

Since $s \geq 1$, we may partition for each $i$ with $s + 1 \leq i \leq r$, the subgraph $H_i \setminus a_i$ into $W_2(H_i)$ complete subgraphs of at most two vertices. When combined with $\{y_1, a_1\}, \{y_2, a_2\}, \ldots, \{y_s, a_s\}$, the construction produces a partition of $G$ into $W_2(G)$ complete subgraphs of at most two vertices.

Anderson [1] uses a similar argument to give an elementary proof of Tutte's factor theorem from Hall's matching theorem.

It is not true that $W_3(G) = P_3(G)$ for all graphs. An example of a poset $X$ for which $W_3(X) < P_3(X)$ is $(3 + 3) + 3$.

6. An extension of Dilworth's embedding theorem. In this section we consider the structure of $2^X$ in more detail in order to make an exact computation of $\text{Dim}_{k2^X}$.

**Theorem 7.** $\text{Dim}_{k2^X} = P_{k-1}(X)$ for all $X$.

**Proof.** Suppose $\text{Dim}_{k2^X} = t$ and let $F : 2^X \rightarrow k^t$ be an embedding. For each $x \in X$ let $f_x : X \rightarrow 2^X$ be defined by $f_x(y) = 0$ if $y \leq x$ in $X$, and $f_x(y) = 1$ otherwise. It follows that $f_x \in 2^X$ for every $x \in X$ and $f_x \neq f_y$ in $2^X$. 

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iff \( x > y \) in \( X \), i.e., the map \( f : \mathbb{R} \to 2^X \) defined by \( f(x) = \{x \} \) is an embedding.

For each \( i \leq t \) let \( X_i = \{ x \in X : y < x \) or \( y \not\in X \} \). Then each \( X_i \) is a chain in \( X \) with \( |X_i| \leq k \). Furthermore if \( |X_i| = k \), then the least element in \( X_i \) is also the least element in \( X \).

We now show that \( X = X_1 \cup X_2 \cup \ldots \cup X_t \). Suppose on the contrary that there exists \( x \in X \) with \( x \not\in X_1 \cup X_2 \cup \ldots \cup X_t \). Then for each \( i \leq t \), there exists a point \( y \in X \) with \( y \not\in X_i \) but \( F(f_x)(i) > F(f_y)(i) \). Let \( C \) be the collection of all subsets \( A \subseteq X \) such that (1) \( a \in A \) implies \( a \not\in x \) and (2) for every \( i \leq t \), there exists \( a \in A \) with \( F(f_x)(i) > F(f_y)(i) \). Now among the sets in \( C \), choose one set say \( A_0 \) with \( |A_0| \) minimum. It follows that \( A_0 \) is an antichain and \( |A_0| \geq 2 \). Now define a function \( f_0 : X \to 2 \) by \( f_0(y) = 0 \) if \( y \leq x \) for some \( a \in A_0 \) and \( f_0(y) = 1 \) otherwise. It follows that \( f_0 \in 2^X \) and \( f_0 < f \) in \( 2^X \) for every \( a \in A_0 \). Furthermore \( f_0 \not\in f_x \) in \( 2^X \) since \( f_0(x) = 1 \) and \( f_x(x) = 0 \). Since \( F \) is an embedding of \( 2^X \) in \( k^t \), there exist \( i \leq t \) with \( F(f_0)(i) > F(f_x)(i) \) and thus \( F(f_a)(i) > F(f_x)(i) \) for every \( a \in A_0 \). The contradiction shows that \( X = X_1 \cup X_2 \cup \ldots \cup X_t \).

If \( X \) has no least element, then \( |X_i| \leq k - 1 \) for all \( i \leq t \) and thus \( P_{k-1}(X) \leq t \). If \( X \) has a least element \( x \), remove \( x \) from each chain in which it appears and let the resulting chains be \( Y_1, Y_2, \ldots, Y_t \). If \( |Y_i| \leq k - 2 \) for some \( i \leq t \), then we conclude that \( P_{k-1}(X) \leq t \).

If \( |Y_i| = k - 1 \) for every \( i \leq t \), then \( F(f_x)(i) = k - 1 \) for every \( i \leq t \). Define \( h : X \to \mathbb{R} \) by \( h(y) = 1 \) for all \( y \in X \). Then \( h > f_x \) in \( 2^X \) but \( F(h) \geq F(h) \) in \( k^t \). The contradiction completes the proof.

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