CODIMENSION OF COMPACT $M$-SEMILATTICES

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ABSTRACT. This paper is a generalization of [5] and gives a partial answer to Question 31 in [1], i.e., if $S$ is a compact $M$-semilattice of finite codimension and $x \neq y$, then there exists a closed subsemilattice $A$ of $S$ such that $A$ separates $x$ and $y$ in $S$ and $\text{cd } A < \text{cd } S$.

A topological semilattice is a Hausdorff space with an associative continuous operation which is commutative and idempotent. We denote

$$M(x) = \{y \in S \mid x \leq y\}, \quad L(x) = \{y \in S \mid y \leq x\}, \quad [x, y] = M(x) \cap L(y),$$

and $F(A)$ the boundary of $A$ in $S$. The breadth of $S$ (Br $S$) is the smallest integer $n$ such that any finite subset $F$ of $S$ contains a subset $G$ of at most $n$ elements such that $\inf F = \inf G$. The codimension of $S$ is the smallest integer $n$ such that $i^*: H^n(S) \rightarrow H^n(A)$ is onto for each closed subset $A$ and inclusion $i$.

I. Lemma 1.1. If $S$ is a topological semilattice and $a \in S$, then $F(M(a))$ is an ideal of $M(a)$ if $F(M(a)) \neq \emptyset$.

Proof. Let $p \in F(M(a))$ and $q \in M(a)$. Suppose $pq$ belongs to the interior of $M(a)$. Then there exists an open set $U$ of $p$ such that $Uq \subseteq M(a)$. For each $z \in U$, $zq \in M(a)$ implies $z \in M(a)$. Hence $p \in U$ which is contained in $M(a)$, contrary to $p \in F(M(a))$.

A topological semilattice $S$ is said to be an $M$-semilattice if $M(x)$ is connected for each $x \in S$.

Lemma 1.2. If $S$ is an $M$-semilattice, then $F(M(a))$ is an $M$-semilattice if nonempty.

Proof. Let $x, y \in F(M(a))$ and $x < y$. Then $M(x)$ is connected. Since $F(M(a))$ is an ideal of $M(a)$, then $yM(x) \subseteq F(M(a)) \cap L(y) \cap M(x)$ and also $yM(x)$ contains $x$ and $y$.

Lemma 1.3. If $S$ is a compact $M$-semilattice, then $S$ is locally connected.

Proof. Let $x \in W$ and $W$ be open in $S$. We can assume $W$ convex since $S$ is a compact partially ordered space. Let $x \in V$ and $V$ open in $S$ and $V^2 \subseteq W$. For all $p, q \in V$, $pq \in V^2$. Then there exist arc-chains $A$ and $B$ from $pq$ to $p$ and $pq$ to $q$ respectively. Since $W$ is convex, then $A$ and $B$ are...
are contained in \( W \). Hence \( S \) is connected in kleinem at each \( x \in S \); \( S \) is locally connected.

**Theorem 1.** If \( S \) is a compact \( M \)-semilattice, then \( F(M(a)) \) is a compact connected locally connected semilattice and if \( S \) has finite codimension, then \( S \) has a basis of closed neighborhoods each of which is a subsemilattice.

**Proof.** Apply Lemma 1.3 and Theorem 3.4 of [2].

**II.** If \( S \) is a compact semilattice with identity, we can define

\[
a \lor b = \inf \{ x \mid a \leq x \text{ and } b \leq x \}
\]

which exists and is the least upper bound for \( a \) and \( b \).

**Lemma 2.1.** Let \( S \) be a compact semilattice and \( \{ x_\alpha \mid \alpha \in D \} \) is an increasing net in \( S \). Then \( x_\alpha \) converges to \( x \) if and only if \( x = \operatorname{lub} \{ x_\alpha \mid \alpha \in D \} \).

**Proof.** Suppose \( x_\alpha \) converges to \( x \). Let \( \beta \in D \). Then \( x_\beta x_\alpha \) converges to \( x_\beta x \). But \( x_\beta \leq x_\alpha \) residually. Hence \( x_\beta x = x_\beta \). Thus \( x_\beta \leq x \), i.e., \( \operatorname{lub} \{ x_\alpha \mid \alpha \in D \} \leq x \). Suppose \( x \notin \operatorname{lub} x_\alpha \). Then there are open sets \( x \in U \), \( \operatorname{lub} x_\alpha \in V \) such that \( (U \times V) \cap \leq = \emptyset \). Choose any \( x_\beta \in U \). Then \( x_\beta \leq \operatorname{lub} x_\alpha \), a contradiction. Hence \( x = \operatorname{lub} x_\alpha \).

Suppose \( x = \operatorname{lub} x_\alpha \). Let \( x \in U \) open in \( S \). Suppose there exists a cofinal subset \( E \) of \( D \) such that \( \{ x_\alpha \mid \alpha \in E \} \cap U = \emptyset \). Then there exists a subnet \( \gamma_\beta \) of \( \{ x_\alpha \mid \alpha \in E \} \) such that \( \gamma_\beta \) converges to some \( y \notin U \). By the previous paragraph, \( y = \operatorname{lub} \gamma_\beta = \operatorname{lub} x_\alpha = x \). We have a contradiction.

**Lemma 2.2.** If \( S \) is a compact semilattice and \( x_\alpha \) is an increasing net converging to \( x \), then \( a \lor x_\alpha \) is an increasing net converging to \( a \lor x \).

**Proof.** Since \( a \lor x_\alpha \leq a \lor x \) for each \( \alpha \), then \( \operatorname{lub}(a \lor x_\alpha) \leq a \lor x \).

Also for each \( \alpha \), \( x_\alpha \leq a \lor x_\alpha \). Hence \( x \leq \operatorname{lub}(a \lor x_\alpha) \) which yields \( a \lor x \leq \operatorname{lub}(a \lor x_\alpha) \).

**Lemma 2.3.** If \( S \) is a compact \( M \)-semilattice with identity, then \( \operatorname{cd} S = \operatorname{Br} S \).

**Proof.** A generalization of Corollary 2.4 of [4].

**Theorem 2.** If \( S \) is a compact \( M \)-semilattice of positive codimension \( n \), then \( \operatorname{cd}(F(M(a))) < n \).

**Proof.** We first show that if \( a \neq x \in F(M(a)) \), then \( \operatorname{cd}[a, x] < n \). Choose a closed neighborhood \( V \) of \( x \) such that \( V \cap L(a) = \emptyset \) and \( V^2 = V \). Since \( x \in F(M(a)) \), then \( U \cap S \setminus M(a) \neq \emptyset \) for each open set \( U \) containing \( x \). Hence there is a net \( \{ x_\alpha \} \subseteq V \) such that \( x_\alpha \notin M(a) \) and \( x_\alpha \) converges to \( x \). Let \( y_\alpha = \inf \{ x_\beta \mid \alpha \leq \beta \} \). Then \( y_\alpha \in V \setminus M(a) \). If \( W \) is a closed neighborhood of \( x \)
and $W^2 = W$, then there exists $\alpha$ such that for each $\beta \geq \alpha$, $x_\beta \in W$. Then $\inf \{ x_\beta | \beta \geq \gamma \} \in W$ for all $\gamma \geq \alpha$. Hence $y_\alpha$ is an increasing net converging to $x$.

For each $\alpha \in D$, $[a, a \lor y_\alpha]$ has breadth less than $n$ [6, Lemma 1.1] since $Br L(x) = cd L(x) \leq n$. Since $\{[a, a \lor y_\alpha]\}$ is a chain of subsemilattices, then

$$Br(\bigcup[a, a \lor y_\alpha]) = Br((\bigcup[a, a \lor y_\alpha])^*) < n,$$

where $*$ denotes closure in $S$. Since $y_\alpha$ converges to $x$, then $a \lor y_\alpha$ converges to $a \lor x$ and $(a \lor y_\alpha)M(a)$ converges to $(a \lor x)M(a)$ in terms of lim inf and lim sup. Since $[a, a \lor y_\alpha] = (a \lor y_\alpha)M(a)$ and $[a, x] = [a, a \lor x] = (a \lor x)M(a)$ and $[a, a \lor y_\alpha] \subseteq [a, a \lor x]$, then $[a, x] = (\bigcup[a, a \lor y_\alpha])^*$.

Thus $Br[a, a] < n$. Since $[a, x]$ is a compact $M$-semilattice with identity, then $cd[a, x] = Br[a, a] < n$.

Suppose $F(M(a)) \neq \emptyset$. Then by Theorem 1 and [3, Corollary 2], $F(M(a))$ has a point $x$ such that $cd(F(M(a)) = cd(F(M(a)) \cap L(x))$. But

$$cd(F(M(a)) \cap L(x)) \leq cd[a, x] < n.$$

**Theorem 3.** If $S$ is a compact $M$-semilattice and $x \neq y$, then there exists a closed subsemilattice $A$ of $S$ such that $A$ separates $x$ and $y$ in $S$ and $cd A < cd S$.

**Proof.** Suppose $x \neq y$. Then $x \not\leq y$ or $y \not\leq x$. Assume $x \not\leq y$. There exist open sets $U, V$ containing $x$ and $y$ such that $(U \times V) \cap \leq = \emptyset$. Let $K$ be a closed neighborhood of $x$ contained in $U$ and $K^2 = K$. Choose $a = \inf K$. Then $x$ belongs to the interior of $M(a)$ and $y \not\in M(a)$. Hence $F(M(a))$ separates $x$ and $y$. Also $cd F(M(a)) < cd S$ by Theorem 2.

**REFERENCES**


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