QUASI-NILPOTENT SETS IN SEMIGROUPS

H. L. CHOW

ABSTRACT. In a compact semigroup $S$ with zero 0, a subset $A$ of $S$ is called quasi-nilpotent if the closed semigroup generated by $A$ contains 0. A probability measure $\mu$ on $S$ is called nilpotent if the sequence $(\mu^n)$ converges to the Dirac measure at 0. It is shown that a probability measure is nilpotent if and only if its support is quasi-nilpotent. Consequently, the set of all nilpotent measures on $S$ is convex and everywhere dense in the set of all probability measures on $S$ and the union of their supports is $S$.

In a topological semigroup with zero 0, an element $x$ is termed nilpotent if $x^n \to 0$ as $n \to \infty$ [5]. This definition has an obvious extension to subsets of the semigroup, i.e. a subset $A$ is nilpotent if $A^n \to 0$ as $n \to \infty$. Now we call a subset $B$ of the semigroup quasi-nilpotent if the closed semigroup generated by $B$ contains the zero 0. It is shown that, when the topological semigroup is compact, a singleton is nilpotent if and only if quasi-nilpotent. Then we investigate the set of probability measures on a compact semigroup and characterize a nilpotent probability measure as a measure with quasi-nilpotent support.

Let $S$ be a topological semigroup with zero 0, and $A$ a subset of $S$. Let $S(A)$ denote the semigroup generated by $A$, i.e. $S(A) = \bigcup_{n=1}^{\infty} A^n$. It is trivial that any subset containing 0 is quasi-nilpotent; in particular, the set $N(S)$ of nilpotent elements of $S$ is quasi-nilpotent. From the semigroup $S$ given in Example 6 below, in which $N(S) = \{0, 1\}$ and $N(S)^n = N(S)$ for all $n$ [4, p. 56], we see that $N(S)$ is not nilpotent.

Theorem 1. Let $A$ be a subset of $S$. Then (i) if $S(A) \cap N(S) \neq \emptyset$ (where the bar denotes closure), then $A$ is quasi-nilpotent.

(ii) If $A^n$ is quasi-nilpotent for some $n$, then $A$ itself is quasi-nilpotent.

Proof. (i) Take $a \in S(A) \cap N(S)$. In view of the fact that $a^n \to 0$, we have $0 \in \overline{S(A)}$, i.e. $A$ is quasi-nilpotent.

(ii) Since $S(A^n) \subset S(A)$ and $0 \in \overline{S(A^n)}$, it follows that $0 \in \overline{S(A)}$, and the theorem is proved.

We remark that, if $A^n$ is nilpotent for some $n$, then $A$ is also nilpotent, by a similar argument to that given in the proof of Lemma 2.1.4 of [4].

Received by the editors April 3, 1974 and, in revised form, July 15, 1974. AMS (MOS) subject classifications (1970). Primary 22A20, 43A05, 60B15; Secondary 22A15.

Key words and phrases. Quasi-nilpotent set, compact semigroup with zero, probability measure, support of a measure, nilpotent measure, nil semigroup.
Evidently a nilpotent set is quasi-nilpotent. As for the converse, which may not be true in general, we prove a special case in

**Theorem 2.** Suppose $S$ is a compact semigroup with $0$. Then $x \in S$ is nilpotent if and only if quasi-nilpotent.

**Proof.** It is enough to show that $x$ is nilpotent if it is quasi-nilpotent. Recall that the minimal ideal $\overline{K(S(x))}$ of the compact semigroup $S(x)$ contains exactly all cluster points of the sequence $(x^n)_{n=1}^{\infty}$ (see, for example, [4, Theorem 3.1.1]). Now $\overline{K(S(x))} = \{0\}$ since $0 \in S(x)$. Thus the sequence $(x^n)$ has a unique cluster point, whence $x^n \to 0$ as $n \to \infty$, completing the proof.

**Remark.** The preceding theorem does not hold for a compact semitopological semigroup (i.e. the multiplication is only separately continuous). For instance, take the compact monothetic semigroup $S(u)$ generated by $u$, with $u$ defined in Example 2 of [1]; then the semigroup has zero $0$ and identity $1$ such that $u^{n!} \to 0$ and $u^n \to 1$. As a consequence, the element $u$ is quasi-nilpotent but not nilpotent.

In what follows $S$ will be a compact semigroup with zero $0$. Denote by $P(S)$ the set of probability measures (i.e. normalized positive regular Borel measures) on $S$. For $\mu, \nu \in P(S)$, define convolution $\mu \ast \nu \in P(S)$ by

$$\int f(z) \, d(\mu \ast \nu)(z) = \iint f(xy) \, d\mu(x) \, d\nu(y)$$

for all continuous functions $f$ on $S$, so that $P(S)$ forms a semigroup. If $P(S)$ is endowed with the weak* topology, i.e. a net $(\mu_n)$ in $P(S)$ converges to $\mu \in P(S)$ if $\int f(x) \, d\mu_n(x) \to \int f(x) \, d\mu(x)$ for continuous functions $f$ on $S$, then $P(S)$ is a compact semigroup [3].

The support of $\mu \in P(S)$, supp $\mu$, is the smallest closed set with $\mu$-mass $1$. It is well known [3, Lemma 2.1] that, for $\mu, \nu \in P(S)$, supp $(\mu \ast \nu) = $ \{supp $\mu \ast \nu$ \}.

Let $\Gamma$ be a subset of $P(S)$ and define its support as the set supp $\Gamma = \bigcup_{\mu \in \Gamma}$ supp $\mu$. It is easy to see that supp $(\Gamma_1 \Gamma_2) = $ \{supp $\Gamma_1 \ast \Gamma_2$ \} for $\Gamma_1 \subset P(S), \Gamma_2 \subset P(S)$.

**Lemma 3.** Let $\Gamma \subset P(S)$. Then supp $S(\Gamma) = \overline{S($supp $\Gamma$)}.$

**Proof.** That supp $S(\Gamma) = \overline{S($supp $\Gamma$)}$ follows from a result in [3, p. 55]. We assert that supp $S(\Gamma) = \overline{S($supp $\Gamma$)}$. Since $S(\Gamma) \supset \Gamma^n$ for $n = 1, 2, \ldots$, clearly supp $S(\Gamma) \supset \overline{S($supp $\Gamma$)}$. And so supp $S(\Gamma) \supset \overline{S($supp $\Gamma$)}$.

Whence supp $S(\Gamma) \supset \overline{S($supp $\Gamma$)}$. On the other hand, take any $\mu \in S(\Gamma)$. Then $\mu \in \Gamma^n$ for some $n$, implying that supp $\mu \subset \overline{S($supp $\Gamma$)}$. This gives supp $S(\Gamma) \subset \overline{S($supp $\Gamma$)}, and the result follows.

Since the Dirac measure $\theta$ at $0$ is a zero in $P(S)$, we can now consider quasi-nilpotent sets in $P(S)$.
Theorem 4. A subset \( \Gamma \subset P(S) \) is quasi-nilpotent if and only if \( \text{supp} \, \Gamma \) is quasi-nilpotent in \( S \).

Proof. Suppose first that \( \Gamma \) is quasi-nilpotent, i.e. \( \theta \in \overline{S(\Gamma)} \). By virtue of Lemma 3, we have \( 0 \in S(\text{supp} \, \Gamma) \) i.e. \( \text{supp} \, \Gamma \) is quasi-nilpotent. Conversely, suppose \( \text{supp} \, \Gamma \) is quasi-nilpotent in \( S \). This means that \( 0 \in S(\text{supp} \, \Gamma) \) and therefore \( \{0\} \) is the minimal ideal \( K(S(\text{supp} \, \Gamma)) \) of the semigroup \( S(\text{supp} \, \Gamma) \). Now consider the minimal ideal \( K(S(\Gamma)) \) of the compact semigroup \( S(\Gamma) \) [6, Theorem 2]. Since \( \text{supp} \, K(S(\Gamma)) = K(\text{supp} \, S(\Gamma)) \) (see, for example, [2, Theorem 5(2)]) and \( \text{supp} \, S(\Gamma) = S(\text{supp} \, \Gamma) \) by Lemma 3, we have \( \{0\} = \text{supp} \, K(S(\Gamma)) \), giving that \( K(S(\Gamma)) = \{\theta\} \). Accordingly \( \theta \in S(\Gamma) \), and the theorem is proved.

By Theorems 2 and 4, we immediately obtain

Theorem 5. A measure \( \mu \in P(S) \) is nilpotent if and only if \( \text{supp} \, \mu \) is quasi-nilpotent in \( S \).

Example 6. The result in Theorem 5 is best possible in the sense that the support of a nilpotent measure in \( P(S) \) need not be a nilpotent subset of \( S \). Take the semigroup \( S = [0, 1] \) with the usual topology and the ordinary multiplication. Let \( \mu \) be the restriction to \( S \) of the Lebesgue measure on the real line. Since \( \text{supp} \, \mu = S \) is quasi-nilpotent, it follows that \( \mu \) is nilpotent. However, \( \text{supp} \, \mu \) is not nilpotent since \( (\text{supp} \, \mu)^n = \text{supp} \, \mu = S \) for all \( n \).

Note that Theorem 5 is not true for the compact semitopological semigroup \( S_w(\mu) \) considered in the Remark above. Obviously the Dirac measure \( \delta(u) \) at \( u \) is not nilpotent while \( \text{supp} \, \delta(u) \) is quasi-nilpotent in \( S \).

Applying Theorem 5, we obtain the following results about the set \( N(P(S)) \) of nilpotent elements in \( P(S) \). First we have a sufficient condition for a probability measure to be nilpotent.

Theorem 7. Let \( \mu \in P(S) \). If \( \text{supp} \, \mu \cap N(S) \neq \emptyset \), then \( \mu \in N(P(S)) \).

Proof. Since \( \overline{S(\text{supp} \, \mu) \cap N(S)} \supset \text{supp} \, \mu \cap N(S) \neq \emptyset \), we see that the set \( \text{supp} \, \mu \) is quasi-nilpotent in \( S \) by Theorem 1 (i). Whence \( \mu \) is nilpotent.

Example 8. The converse of Theorem 7 may not hold. For instance, take the semigroup \( S \) with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>c</td>
</tr>
</tbody>
</table>

Then the measure \( \mu \in L(S(a), S(b)) \cap N(P(S)) \) since \( 0 \in \text{supp} \, \mu^2 \). However, \( \text{supp} \, \mu \cap N(S) = \{b, c\} \cap \{0, a\} = \emptyset \).
Corollary 9. (i) \( N(P(S)) \) is a noncountable set.
(ii) \( \bigcup \{ \text{supp } \mu : \mu \in N(P(S)) \} = S \).

Proof. (i) Take any measure \( \mu \neq \emptyset \) and real number \( 0 \leq t < 1 \). Then the measure \( t\mu + (1 - t)\emptyset \) is nilpotent since \( 0 \in \text{supp } (t\mu + (1 - t)\emptyset) \cap N(S) \). Hence the set \( N(P(S)) \supset \{ t\mu + (1 - t)\emptyset : 0 \leq t < 1 \} \) and so is noncountable.

(ii) Let \( a \in S \). Since \( 0 \in \text{supp } \frac{1}{2}(\delta(a) + \emptyset) \cap N(S) \), it follows that \( \frac{1}{2}(\delta(a) + \emptyset) \in N(P(S)) \). That \( a \in \text{supp } \frac{1}{2}(\delta(a) + \emptyset) \) gives the result.

A semigroup with zero is said to be nil if each element is nilpotent.

Theorem 10. \( P(S) \) is nil if and only if \( S \) is nil.

Proof. The "if" part follows from the fact that, for \( \mu \in P(S) \), \( \text{supp } \mu \cap N(S) = \text{supp } \mu \neq \emptyset \). To prove the "only if" part, take \( a \in S \) and note that \( \delta(a) \) is nilpotent in \( P(S) \). So \( a \) is nilpotent in \( S \) and the proof is complete.

Lemma 11. Let \( \mu, \nu \in P(S) \). If \( \mu \in N(P(S)) \) and \( \text{supp } \mu \subset \text{supp } \nu \), then \( \nu \in N(P(S)) \).

Proof. This is immediate since \( 0 \in S(\text{supp } \mu) \subset S(\text{supp } \nu) \).

Theorem 12. (i) \( N(P(S)) \) is a convex set and hence connected.
(ii) \( N(P(S)) = P(S) \).

Proof. (i) Take \( \mu, \nu \in N(P(S)) \). For real number \( 0 < t < 1 \), the measure \( t\mu + (1 - t)\nu \in N(P(S)) \) since
\[
\text{supp } (t\mu + (1 - t)\nu) = \text{supp } \mu \cup \text{supp } \nu \supset \text{supp } \mu.
\]
Thus \( N(P(S)) \) is convex.

(ii) Let \( \tau \in P(S) \). Clearly \( \theta/n + (n - 1)\tau/n \in N(P(S)) \) for any positive integer \( n \). As the sequence \( (\theta/n + (n - 1)\tau/n)_{n=1}^{\infty} \) converges to \( \tau \), we see that \( N(P(S)) \) is dense in \( P(S) \).

Corollary 13. Let \( W \) be a subset of \( P(S) \). If \( W \supset N(P(S)) \), then \( W \) is a connected set.

Proof. This follows simply from the previous theorem.

For any \( \mu \in P(S) \), it is a well-known fact that the sequence \( ((\mu + \mu^2 + \cdots + \mu^n)/n)_{n=1}^{\infty} \) must converge to a measure \( L(\mu) \in P(S) \) such that \( \text{supp } L(\mu) \) is the minimal ideal of the semigroup \( S(\text{supp } \mu) \); see [7] or [8].

Theorem 14. The measure \( \mu \in P(S) \) is nilpotent if and only if \( L(\mu) = \emptyset \).

Proof. In view of the fact that \( L(\emptyset) = \emptyset \) if and only if \( S(\text{supp } \mu) \) contains \( 0 \), we apply Theorem 5 to conclude the proof.
REFERENCES


DEPARTMENT OF MATHEMATICS, CHUNG CHI COLLEGE, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG