RELATIVELY UNIFORM BANACH LATTICES

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ABSTRACT. Sequential relative uniform and norm convergence agree in a Banach lattice, if and only if it is equivalent to an $M$-space.

Let $(E, \leq, \| \cdot \|)$ be a Banach lattice. The sequence $\{f_n\}$ is said to be relatively uniformly (abbreviated ru) convergent to $f$, if there exists $u > 0, \lambda_n \in \mathbb{R}$, $\lambda_n \neq 0$, such that $|f_n - f| \leq \lambda_n u$ for all $n$. If there exist an increasing sequence $\{g_n\}$ and a decreasing sequence $\{h_n\}$ such that $g_n \leq f_n \leq h_n$ for all $n$, and $\bigvee g_n = \bigwedge h_n = f$, we say $\{f_n\}$ is order (abbreviated o) convergent to $f$. We say two Banach lattices are equivalent if there exists a vector lattice isomorphism under which the norms are equivalent. Call $E$ relatively uniform if norm and ru convergence agree, for sequences, on $E$.

Firstly we indicate why the corresponding notion for nets is rather trivial. We say the net $\{f_\alpha\}_{\alpha \in A}$ ru converges to $f$ if there exists $u > 0$, such that given any $\epsilon > 0$ there exists $\alpha_0 \in A$ with $|f_\alpha - f| \leq \epsilon u$ for all $\alpha \geq \alpha_0$.

Proposition 1. $E$ is net relatively uniform if and only if $E$ contains a strong unit.

Proof. Suppose $E$ is net relatively uniform. If $\alpha, \beta \in E \setminus \{0\}$ write $\alpha \approx \beta$ to mean $\|\alpha\| \geq \|\beta\|$. Let $f_\alpha = \alpha$, then $\|\cdot\| - \text{lim} f_\alpha = 0$, so ru-\text{lim} $f_\alpha = 0$. Hence $E$ contains a strong unit.

Conversely suppose $E$ contains a strong unit $e$. The norms $\| \cdot \|$ and $\| \cdot \|_e$ are equivalent, by Birkhoff [1, Lemma 5, p. 367], where $\| \|_e = \inf \lambda: |f| \leq \lambda e$. $\square$

From now on we only consider sequential convergence.

The following result is contained, implicitly, in Leader [3].

Lemma 1 (Leader). If $E$ is relatively uniform then it is equivalent to an $M$-space.

Proof. By [3, Theorem 4] it follows that $\| \cdot \|$ is equivalent to an $M$ norm if and only if, for every sequence $\{p(n)\}$ of positive integers, and for every sequence $\{f_n\}$ such that $\|f_n\| \to 0$, we have

$$\|f_n\| \vee |f_{n+1}| \vee \cdots \vee |f_{n+p(n)}| \to 0.$$
If $E$ is relatively uniform and $\|f_n\| \to 0$, then $|f_n| \vee |f_{n+1}| \vee \cdots \vee |f_{n+p(n)}| \leq \lambda_n u$, hence (1) holds. □

**Lemma 2.** In an $M$ space a norm convergent sequence has a supremum.

**Proof.** Let $\{f_n\}$ be a norm convergent sequence. Denote $f_1 \vee \cdots \vee f_n$ by $g_n$. We show $\{g_n\}$ is Cauchy. Let $n > m$, then

$$g_n - g_m = g_m \vee f_{m+1} \vee \cdots \vee f_n - g_m = (f_{m+1} - g_m)^+ \vee \cdots \vee (f_n - g_m)^+.$$ 

Now

$$f_i - g_m = f_i - f_1 \vee \cdots \vee f_m = (f_i - f_1)^+ \vee \cdots \vee (f_i - f_m)^+,$$

so $(f_i - g_m)^+ \leq (f_i - f_m)^+$. Hence

$$\|g_n - g_m\| = \left\| \vee_{i=m+1}^{n} (f_i - g_m)^+ \right\| \leq \left\| \vee_{i=m+1}^{n} (f_i - f_m)^+ \right\| = \vee_{i=m+1}^{n} \|f_i - f_m\|.$$

Thus $\{g_n\}$ is Cauchy. Let $\|\cdot\| \lim g_n = g$; it then follows, from the fact that the positive cone is (norm) closed, that $g = \vee_{n=1}^{\infty} f_n$. □

**Proposition 2.** The following statements are pairwise equivalent:

(i) $E$ is relatively uniform;

(ii) $E$ is equivalent to an $M$ space;

(iii) $\{f_n\}$ norm converges implies that $\vee f_n$ exists;

(iv) $\{f_n\}$ norm converges implies that $\wedge f_n$ exists;

(v) $\{f_n\}$ norm converges implies that $\{f_n\}$ order converges.

**Proof.** (i) implies (ii) is Lemma 1, (ii) implies (iii) is Lemma 2.

(iii) implies (v). If $\|f_n\| \to 0$ then $u = \vee_{n=0}^{\|f_n\|} \|f_n\|^{-\frac{1}{2}} |f_n|$ exists. So $f_n u$ converges to 0. Since $E$ is archimedean, (v) follows.

(iii) and (iv) are equivalent. If $\{f_n\}$ norm converges then so does $\{-f_n\}$, and $\vee -f_n = -\wedge f_n$.

(v) implies (i). Suppose $\|f_n\| \to 0$ then $\|f_n\|^{-\frac{1}{2}} |f_n|$ norm converges to 0, so by (v) there exists $v \geq \|f_n\|^{-\frac{1}{2}} |f_n|$, hence (i) holds (we may assume without loss of generality that $\|f_n\| > 0$ for all $n$). □

A Banach lattice is called an $L^p$ space if $|f| \wedge |g| = 0$ implies that $\|f + g\| = (\|f\|^p + \|g\|^p)^{1/p}$ where $1 \leq p < \infty$ (Birkhoff [1, p. 378]).

**Corollary 1.** An infinite dimensional $L^p$ space is not relatively uniform, and is not equivalent to an $M$ space.
Proof. Suppose there exists an infinite dimensional relatively uniform $L^p$ space, $E$. By Zaanen [4, Theorem 4.3] there exists a sequence $\{f_n\}$ of strictly positive pairwise orthogonal elements in $E$. We may assume that $\|f_n\| = n^{-1/p}$. So $\nu\cdot\lim f_n = 0$, and hence there exists $u \in E$ such that $f_n \leq u$ for all $n$. So

$$\|x\| \geq \|f_1 \vee \cdots \vee f_n\| = \|f_1 + \cdots + f_n\|$$

$$= (\|f_1\|^p + \cdots + \|f_n\|^p)^{1/p} = \left(\sum_{k=1}^{n} \frac{1}{k}\right)^{1/p} \to \infty,$$

giving the required contradiction. The rest follows from Proposition 2. □

Corollary 2. In an infinite dimensional $L^p$ space every $l$-ideal, generated by countably many elements, is proper.

Proof. Suppose $E$ is an infinite dimensional $L^p$ space, containing a sequence $\{f_n > 0\}$ which is contained in no proper $l$-ideal. Let $e = \sum_{n=1}^{\infty} n^{-2} \|f_n\|^{-1} f_n$, then $e$ is a strong unit. With the equivalent norm $\|\cdot\|_e$ (see Proposition 1) $E$ becomes an $M$ space, with unity $e$. The rest follows by Corollary 1. □

In the case of $l^1$ the above reduces to the well-known result, that even with countably many 'test' series the comparison test is not effective on all positive term series; see Knopp [2].

REFERENCES


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