ON BIESTERFELDT'S COMPLETION AXIOM SPACES

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ABSTRACT. It is proved that a Hausdorff, totally bounded Completion Axiom space is a uniform space. The method of proof shows that Hausdorff Completion Axiom spaces have completions (in the embedding sense) which are again Hausdorff Completion Axiom spaces; moreover these completions are uniformly regular, uniformly strict, and have the regular extension property.

1. Introduction. The first uniform convergence spaces [2] to be completed, in the embedding sense, were the Completion Axiom (hereafter abbreviated C.A.) spaces of Biesterfeldt [1]. Later Richardson showed that indeed any uniform convergence space has a completion [7]; and since then various papers on completions of uniform convergence spaces or their Cauchy structures have appeared. (See, for example [4] or [5].)

However, the extent to which these original C.A. spaces deviate from uniform spaces is still not determined. The only known result is that a C.A. space has a regular topology as its induced convergence structure [6]. In this note it will be shown that a Hausdorff, totally bounded C.A. space is a uniform space. In doing so, we will prove that Hausdorff C.A. spaces have very strong (in a sense to be made precise below) completions, with extension property, within their own category.

The reader is referred to [2] and [3] for basic definitions although a few pertinent ones are given here. If \((X, I)\) is a uniform convergence space (u.c.s.), a \(\Delta\)-symmetric base for \(I\) is a base consisting of symmetric filters, each of which is coarser than the diagonal filter. Then \((X, I)\) is a C.A. space if there exists a \(\Delta\)-symmetric base \(B\) for \(I\) such that \(\mathcal{F}\) is Cauchy in \(X\) iff \(\mathcal{F} \times \mathcal{F} \geq \Phi\) for every \(\Phi \in B\). \((X, I)\) is \emph{uniformly regular} if \(\text{cl}_{X \times X} \Phi \in I\) whenever \(\Phi \in I\). Completions are taken in the embedding sense; that is \((X^*, I^*)\) is a completion of \((X, I)\) if there is a uniform homeomorphism of \((X, I)\) onto a dense subspace of \((X^*, I^*)\) and \((X^*, I^*)\) is complete. A completion \((X^*, I^*)\) of \((X, I)\) is \emph{uniformly strict} if there is some \(\Phi^* \in I^*\) such that when \(\forall^* \in \Phi^*\) and \(y \in X^*, V^*(y) \cap X \neq \emptyset\).
2. Results.

Proposition 1. Let (X, l) be a u.c.s. with A-symmetric base $\beta$. Then, for $A \subseteq X \times X$, $\text{cl}_{X \times X} A = \bigcup (V \circ A \circ V : V \in \Phi)$, the union being taken over all $\Phi \in \beta$.

Proof. Notice first that for any u.c.s. $(X, l)$ with A-symmetric base $\beta$ and $B \subseteq X$, $\text{cl}_X B = \bigcup (B(\Phi) : \Phi \in \beta)$ where $B(\Phi) = \{x : V(x) \cap B \neq \emptyset \text{ for all } V \in \Phi\}$.

Now let $\beta^*$ be any A-symmetric base for the product u.c.s. and let $(x, y) \in \text{cl}_{X \times X} A$. By applying the remark above to $\beta^*$, it follows that $(x, y) \in A(\Phi^*)$ for some $\Phi^* \in \beta^*$. If $p_1$, $p_2$ are the projection maps it is seen that $(p_1 \times p_2)\Phi^* \circ (p_2 \times p_2)\Phi^*$ exists and is therefore finer than some $\Psi \in \beta$. We assert that $(x, y) \in \bigcap (V \circ A \circ V : V \in \Psi)$. For if $V \in \Psi$ and $(p_1 \times p_1)V \circ (p_2 \times p_2)V \subseteq V$, $V$ symmetric in $\Phi^*$, then $V(x, y) \cap A$ is not empty so there exists $(a_1, a_2) \in A$ with $((a_1, a_2), (x, y)) \in V$. Thus $(x, y) \in (p_1 \times p_1)V \circ A \circ (p_2 \times p_2)V \subseteq V \circ A \circ V$.

Conversely, assume $(x, y) \in \bigcap (V \circ A \circ V : V \in \Phi)$ for some $\Phi \in \beta$. Then there are points $(x, a(V)) \in V$, $(a(V), b(V)) \in A$, $(b(V), y) \in V$. The filters of sections of the nets $(a(V) : V \in \Phi)$, $(b(V) : V \in \Phi)$, converge, respectively, to $x$, $y$. Hence the filter of sections of the net $((a(V), b(V)) : V \in \Phi)$ converges to $(x, y)$ and contains $A$. This means $(x, y) \in \text{cl}_{X \times X} A$.

Proposition 2. Let $(X, l)$ have a uniformly regular, uniformly strict completion $(X^*, l^*)$. Then $l^*$ is generated by filters of the form $\text{cl}_{X^* \times X^*} \Phi$, $\Phi \in l$.

Proof. First, if $\Phi \in l$, $\text{cl}_{X^* \times X^*} \Phi \in l^*$ follows from the uniform regularity of $(X^*, l^*)$.

Let $\Phi^* \in l^*$ and put $\Phi_1^* = \Phi^* \cap \Psi^*$ where $\Psi^*(x) \cap X \neq \emptyset$ for all $x \in X^*$, $V^* \in \Psi^*$. With no loss of generality assume $\Phi^*_1$ is symmetric and coarser than the diagonal filter. Define $\Sigma^* = \Phi_1^* \circ \Phi_1^* \circ \Phi_1^*$; we will show that $\Phi^*_1 \geq \text{cl}_{X^* \times X^*} \Sigma_1$, where $\Sigma_1$ is the restriction of $\Sigma^*$ to $X \times X$. Let $C \in \Sigma_1$, $C = C^* \cap X \times X$, $C^* \in \Sigma^*$. Choose a symmetric $V_1^* \in \Phi_1^*$ such that $V_1^* \circ V_1^* \circ V_1^* \subseteq C^*$. Let $(x, y) \in V_1^*$. To show that $(x, y) \in \text{cl}_{X^* \times X^*} C$, it suffices, by Proposition 1, to prove that $(x, y) \in \bigcap (V^* \circ C \circ V^* : V^* \in \Phi_1^*, V^* \text{ symmetric}, V^* \subseteq V_1^*)$.

Let $V^*$ be as indicated; then $V^* \in \Psi^*$ so there are points $z_1 \in V^*(x) \cap X$, $z_2 \in V^*(y) \cap X$. Thus $(z_1, z_2) \in V^* \circ V^* \circ V^* \subseteq C^*$. Also $(z_1, z_2) \in C$ because $z_1$, $z_2 \in X$. Hence $(x, y) \in V^* \circ C \circ V^*$ as desired.

Proposition 3. Let $(X, l)$ be uniformly regular with uniformly strict, uniformly regular completion $(X^*, l^*)$. If $(Y, D)$ is any complete, uniformly regular u.c.s. and $f : (X, l) \rightarrow (Y, D)$ is uniformly continuous, then $f$ has a uniformly continuous extension $f^* : (X^*, l^*) \rightarrow (Y, D)$.
Proof. Define $f^*$ by $f^*(y) = \lim f(y)$ where $\mathcal{F} \to y$ in $X^*$ and $X \in \mathcal{F}$. Since the embedding $X \to X^*$ is also an embedding of the Cauchy structures, $f^*$ is well defined. Let $\Phi^* \in I^*$. By Proposition 2, $\Phi^* \geq cl_{X^* \times X^*} \Phi$ for some $\Phi \in I$. Since $(Y, D)$ is uniformly regular, $cl_{Y \times Y}(f \times f)\Phi \in D$, so to prove that $f^*$ is uniformly continuous it suffices to show that $(f^* \times f^*)(cl_{X^* \times X^*} \Phi) \geq cl_{Y \times Y}(f \times f)\Phi$. We claim, in fact, that, if $A \in \Phi$, $(f^* \times f^*)(cl_{X^* \times X^*} A) \subset cl_{Y \times Y}(f \times f)A$. For if $(x, y) \in cl_{X^* \times X^*} A$, then $(x, y) \in \bigcap (V^* \circ A \circ V^* : V^* \in \Phi^*)$ for some $\Phi^* \in I^*$ by Proposition 1. As in the last part of the proof of Proposition 1, there are nets $a(V^*) \to x$, $b(V^*) \to y$ with the section filter $\mathcal{G}$ of the product net containing $A$. Then

$$(f^* \times f^*)(x, y) = \lim (f \times f)_{\mathcal{G}} \in cl_{Y \times Y}(f \times f)A.$$  

Remark. The referee has made the observation that, for C.A. spaces, the completion of Theorem 1 below is that of Biesterfeldt [1] when equivalent Cauchy filters are identified. Moreover, he observes that if $U(\langle \mathcal{F} \rangle)$ is chosen to be an ultrafilter in $(\mathcal{F})$, then the completion of Theorem 1 is the Kowalsky completion. (See [5, p. 103].)

Theorem 1. Let $(X, I)$ be a Hausdorff C.A. space. Then $(X, I)$ has a uniformly regular, uniformly strict, Hausdorff completion which also is a C.A. space. This completion has the extension property relative to uniformly continuous maps and complete, uniformly regular u.c.s.

Proof. Let $X^\wedge$ be the collection of equivalence classes $(\mathcal{F})$ of Cauchy filters on $X$ under the relation $\mathcal{F} \sim \mathcal{G}$ if $\mathcal{F} \times \mathcal{G}$ is in $I$. For each $(\mathcal{F})$ let $U(\langle \mathcal{F} \rangle)$ be an arbitrary Cauchy filter equivalent to $\mathcal{F}$ with the understanding that $U(\langle \mathcal{F} \rangle) = \mathcal{F}$. For $S \subset X \times X$ let $S^\wedge$ be the collection of all pairs $(\langle \mathcal{F} \rangle, \langle \mathcal{G} \rangle)$ such that $A \times B \subset S$ for some $A, B$ in $U(\langle \mathcal{F} \rangle), U(\langle \mathcal{G} \rangle)$ respectively. Let $\beta$ be an admissible $\Delta$-symmetric base for the C.A. space $(X, I)$. For each $\Phi \in \beta$ define $\Phi^*$ to be the filter generated by the $A^\wedge, A \in \Phi$. If $A \in \Phi, \Phi \in \beta$ and $\mathcal{F}$ is any Cauchy filter on $X$, then $\mathcal{F} \times \mathcal{F} \geq \Phi$ by C.A. Hence $(\langle \mathcal{F} \rangle, \langle \mathcal{G} \rangle) \in A^\wedge$ and each $\Phi^*$ is coarser than the diagonal filter in $X^\wedge \times X^\wedge$. Standard arguments show, then, that the collection $\beta^\wedge$ of all $\Phi^*, \Phi \in \beta$ is a $\Delta$-symmetric base for a u.c.s. $I^\wedge$ on $X^\wedge$. Define $j: X \to X^\wedge$ by $j(x) = \langle x \rangle$. Notice that if $\Phi \in \beta$ and $\langle \mathcal{F} \rangle \in X^\wedge, A \in \Phi$, then $U(\langle \mathcal{F} \rangle) \times U(\langle \mathcal{F} \rangle) \geq \Phi$ by C.A. so $U \times U \subset A$ for some $U \in U(\langle \mathcal{F} \rangle)$. Thus whenever $y \in U, \langle y \rangle \in A^\wedge(\langle \mathcal{F} \rangle)$. This shows that $(X^\wedge, I^\wedge)$ is uniformly strict if, in fact, it is a completion. $(X^\wedge, I^\wedge)$ is complete: Suppose $\Omega \times \Omega \geq \Phi^*$ for $\Phi \in \beta$ and define, for $T \in \Omega$, $A \in \Phi, [A, T] = \{x: \langle x \rangle \in A^\wedge(T)\}$. The preceding paragraph shows that $[A, T] \neq \emptyset$ and it follows easily that the collection $\mathcal{F}$ of supersets of the $[A, T]$ is a filter on $X$. In fact $\mathcal{F}$ is a Cauchy filter. For if $A \in \Phi$ there exists $T \in \Omega$ such that $T, T \subset A$. If $x \in \langle \mathcal{F} \rangle \cap (A \times T)$ then there are filters $\mathcal{G}$, $\mathcal{H}$ in $T$ such that $(\langle \mathcal{G} \rangle, \langle \mathcal{H} \rangle) \in \mathcal{F}$, $(\langle y \rangle, \langle \mathcal{H} \rangle) \in A^\wedge$. Hence $(\langle x \rangle, \langle y \rangle) \in A^\wedge \circ
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A^n o A^n C (A o A o A)^n or (x, y) e A^3. This means \( \mathcal{F} \times \mathcal{F} \geq \Phi^3 \) so \( \mathcal{F} \) is Cauchy. It is now asserted that \( \Omega \rightarrow (\mathcal{F}) \), in fact that \( \Omega \times (\mathcal{F}) \geq \Phi^\omega \). If \( A \in \Phi \) then \( \mathcal{F} \times \mathcal{F} \geq \Phi \) by C.A. so \( [A_1, T] \times [A_1, T] \subset A \) for some \( T \in \Omega, A_1 \in \Phi \), \( A_1 \subset A \). Let \( (\mathcal{O}) \in T \); using C.A. again, \( U \times U \subset A_1 \) for some \( U \in \mathcal{U}(\mathcal{O}) \) hence \( U \subset (x, \langle \delta \rangle) \subset A_1(\mathcal{O}) \subset [A_1, T] \). Thus \( U \times [A_1, T] \subset A \), hence \((\mathcal{O}), (\mathcal{F}) \) is \( A^\omega \). That is, \( T \times (\mathcal{F}) \subset A^\omega \).

The fact that \( j \) is a uniform homeomorphism follows from the relation \((j \times j)\Phi \geq \Phi^\omega \) and the fact that \((j \times j)\Psi \geq \Phi^\omega \) implies \( \Psi \geq \Phi \). It is obvious that \((X^\omega, I^\omega) \) is \( T_1 \), hence Hausdorff.

Notice that any C.A. space \((X, I)\) is uniformly regular because \( \text{cl}_{X \times X} \Phi \geq \Phi^3 \) by Proposition 2 of [6]. Thus the proof of this theorem is complete if it can be shown that \((X^\omega, I^\omega) \) is a C.A. space. For this suppose \( \Omega \) is Cauchy in \( X^\omega \), \( \Omega \times \Omega \geq \Phi^\omega \), and let \( \Phi_1 \) be any member of \( B \). The \( F \) constructed in the proof of completeness is Cauchy in \( X \) so, by C.A. in \( X \), \( \mathcal{F} \times \mathcal{F} \geq \Phi_1 \). Hence, to show \( \Omega \times \Omega \geq (\mathcal{F} \times \mathcal{F})^\omega \) it suffices to prove \( \Omega \times \Omega \geq (\mathcal{F} \times \mathcal{F})^\omega \). For this let \([A, T]\) be given in \( F \); we may assume \( T_1 \times T_1 \subset A^\omega \) for some \( T_1 \in \Omega, T_1 \subset T \). Then \( T_1 \times T_1 \subset ([A, T_1] \times [A, T_1])^\omega \), for if \( (\mathcal{O}) \), \( (\mathcal{H}) \in T_1 \), there are \( U \in \mathcal{U}(\mathcal{O}), V \in \mathcal{U}(\mathcal{H}) \) such that \( U \times U \subset A, V \times V \subset A \) (C.A. has been used again) so \( U \subset (x, \langle \delta \rangle) \subset A^\omega(\mathcal{O}) \subset [A, T_1] \) and \( V \subset (x, \langle \delta \rangle) \subset A^\omega(\mathcal{H}) \subset [A, T_1], \) thus \((\mathcal{O}), (\mathcal{H}) \in ([A, T_1] \times [A, T_1])^\omega \).

We note that the extension property of \((X^\omega, I^\omega) \) and the fact that a uniform space is uniformly regular implies that \((X^\omega, I^\omega) \) is the usual completion of \((X, I) \) whenever \( I \) is a Hausdorff uniformity.

**Proposition 4.** Let \((X, I)\) be a C.A. space. Then \((X, I)\) is totally bounded if and only if there exists \( \Phi \in I \) such that whenever \( V \in \Phi \), there is a finite subset \( F \) of \( X \) such that \( V(F) = X \).

**Proof.** Suppose \((X, I)\) is totally bounded and let \( \Phi \) be a member of an admissible \( \Delta \)-symmetric base for \( I \). Suppose for some \( V \in \Phi \), \( |X - V(F)| > F \) is finite \( F \) is a filter base on \( X \). If this filter base is coarser than a Cauchy filter \( \mathcal{F} \), then \( \mathcal{F} \times \mathcal{F} \geq \Phi \) by C.A. A contradiction now is obtained exactly like the uniform space case. Conversely suppose the condition of the proposition holds and let \( \mathcal{U} \) be an ultrafilter on \( X \). If \( V \in \Phi \) is symmetric, some \( V(x) \in \mathcal{U} \) and \( V(x) \times V(x) \subset V^2 \), so \( \mathcal{U} \times \mathcal{U} \geq \Phi^2 \) and \( \mathcal{U} \) is Cauchy.

**Theorem 2.** A totally bounded Hausdorff C.A. space is a uniform space.

**Proof.** Let \((X^\omega, I^\omega)\) be the completion of Theorem 1. By the proof of strictness of \((X^\omega, I^\omega)\), \( \Phi \in B \) implies \( V^\omega(y) \cap j(x) \neq \emptyset \) for each \( y \in X^\omega \), \( V \in \Phi \). Hence, by Proposition 4, total boundedness of \((X, I)\) carries over to \((X^\omega, I^\omega)\). So the structure induced by \( I^\omega \) is a compact Hausdorff topology.

But there is only one possible C.A. structure which induces a compact, Hausdorff topology, namely a uniformity [6, Proposition 2]. Since \((X, I)\) is
embedded uniformly in \((X^\sim, I^\sim)\), \((X, I)\) is a uniform space.

**Corollary 1.** Let \((X, I)\) be a Hausdorff C.A. space. Then \(I\) induces a completely regular topology on \(X\) if and only if there is a totally bounded, Hausdorff C.A. structure on \(X\) which induces the same topology as \(I\).

Whether each C.A. space is a uniform space remains unresolved. A counterexample would produce a proper class of u.c.s. strictly larger than the class of uniform spaces which has completions within its own category.

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**REFERENCES**


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