A NOTE ON THE TOPOLOGY OF C-CONVERGENCE IN HYPERSPACES

PEDRO MORALES

ABSTRACT. In this note we generalize and partially correct a recent Tychonoff theorem for hyperspaces of F. A. Chimenti [1].

For a topological space $X$, the symbols $\exp^*(X)$, $[\exp^*(X)]$ will denote the hyperspace of all nonempty subsets, of all nonempty closed subsets, respectively, of $X$. In [1, p. 284], F. A. Chimenti claims the following result:

Theorem A. If $\exp^*(X_i)$ is equipped with a topology that preserves the C_i-convergence for every $i \in I$, then the product space $\prod_{i \in I} \exp^*(X_i)$ is compact if and only if the $X_i$ are compact.

The necessity part of Theorem A is not true, as is seen by choosing the $X_i$ noncompact and assigning to each $\exp^*(X_i)$ the indiscrete topology. The purpose of this note is to generalize the sufficiency part of Theorem A and to give a corrected version of the necessity part.

In [1, p. 283] it is shown that there exist nonindiscrete topologies on $\exp^*(X)$ preserving C-convergence. It is clear that there exists a largest topology, denoted $T_C$, on $\exp^*(X)$ preserving C-convergence. We will say that a subset $F$ of $\exp^*(X)$ is C-closed if no net in $F$ C-converges to an element of $\exp^*(X) - F$. It is obvious that the set of all C-closed subsets of $\exp^*(X)$ defines a topology $T$ on $\exp^*(X)$ such that a subset of $\exp^*(X)$ is $T$-closed if and only if it is C-closed. The lower semifinite topology $T_L$ on $\exp^*(X)$ is the topology having as open subbase the subsets of $\exp^*(X)$ of the form $\{A: A \cap U \neq \emptyset\}$, where $U$ is open in $X$ [3, p. 179]. It is clear that $T_L$ preserves C-convergence, that is, $T_L \subseteq T_C$. Of the following four properties, only the last requires a formal proof, in which case, we apply the argument of Theorem 4.2 of [3, p. 161]:

1. $T_C = T_C$. In fact, it suffices to note that $T_C$ preserves C-convergence.

2. If $\{\exp^*(X)\} \subseteq F \subseteq \exp^*(X)$, then the topology induced on $F$ by $T_C$ is the largest topology on $F$ preserving C-convergence.
(3) If \( X \) is compact and \( \text{exp}^*(X) \subseteq F \subseteq \text{exp}^*(X) \), then \( F \) is \( T_C \)-compact. In fact, it suffices to note that \( F \) is \( C \)-compact, since \( \text{exp}^*(X) \) is \( C \)-compact [1, p. 282].

(4) If \( \text{exp}^*(X) \subseteq F \subseteq \text{exp}^*(X) \) and \( F \) is \( T_L \)-compact, then \( X \) is compact. In fact, let \( \{U_i\}_{i \in I} \) be an open cover of \( X \). Write \( \{U_i\} = \{A : A \in F \text{ and } A \cap U_i \neq \emptyset\} \). Then \( \{\{U_i\}_{i \in I} \} \) is an open cover of \( F \), and so contains a finite subcover \( \{\{U_{i_k}\}_{1 \leq k \leq n}\} \) of \( F \). Let \( x \in X \). Then \( \{x\} \subseteq F \), so \( \{x\} \subseteq \{U_{i_k}\} \) for some \( k \), that is, \( x \in U_{i_k} \).

Properties (3) and (4), together with the classical Tychonoff theorem, yield

**Theorem.** For each \( i \in I \), let \( \text{exp}^*(X_i) \subseteq F_i \subseteq \text{exp}^*(X_i) \) and let \( T_i \) be a topology on \( F_i \). Then:

(a) If \( T_i \subseteq T_C \) and \( X_i \) is compact for all \( i \in I \), then the product space \( \prod_{i \in I} F_i \) is compact.

(b) If \( T_L \subseteq T_i \) for all \( i \in I \) and the product space \( \prod_{i \in I} F_i \) is compact, then the \( X_i \) are compact.

**Remarks.** (i) Under the additional hypothesis \( T_L \subseteq T_i \) for all \( i \in I \), the conclusion of Theorem A is true. But in this case, our Theorem yields a larger class of spaces for which the same conclusion holds.

(ii) The final remark of [1] asserts that if \( \text{exp}^*(X_i) \) is equipped with a topology that preserves the \( C_i \)-convergence and the \( X_i \) are \( T_1 \) compact, then the product space \( \prod_{i \in I} \text{exp}^*(X_i) \) is compact. The Theorem contains this result without the \( T_1 \) restriction.

(iii) For each \( i \in I \), let \( T_i \) be a topology of finite type on \( F_i \) [1, p. 283]. Then \( T_L \subseteq T_i \) and, if \( F_i \) is a set of compact subsets of \( X_i \), then \( T_i \subseteq T_C \). The Theorem applies to this case. In particular, if \( T_i \) is the Vietoris topology, we obtain Theorem 3.3 of [2] with its converse.

**REFERENCES**


DEPARTEMENT DE MATHEMATIQUES, UNIVERSITE DE MONTRAL, MONTRAL, QUEBEC, CANADA