R₃-quasi-uniform spaces
and topological homeomorphism groups

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ABSTRACT. It is well known that if X is a completely regular space and G is a homeomorphism group of X onto itself such that G is equicontinuous with respect to a compatible uniformity of X, then G is a topological group under the topology of pointwise convergence. In this paper, we obtain a generalization of the above result by means of R₃-quasi-uniformities.

1. Introduction. Let (X, r) be a topological space. It is well known that if U is a compatible uniformity on X such that G is a homeomorphism group that is equicontinuous with respect to U, then G is a topological group under the topology of pointwise convergence. R. V. Fuller has obtained an analogous result for regular spaces [2] and we have shown previously that a similar result applies when (X, r) is only an R₀ space (and hence, in particular, if X is T₁ or regular) [6]. In this paper we use R₃-quasi-uniformities to complement Fuller’s result. We take the domain space (X, r) to be an arbitrary topological space. If our domain space is regular, it is known that there exists a compatible R₃-quasi-uniformity U on X, that is, r = τₚ [5]. Finally we give a simple example of a non-R₀ topological space (hence not regular) for which our principal result, Theorem 6, obtains.

Let Y be a topological space. A collection U* of two-element open covers of Y is said to be a semiuniformity for Y if for each q ∈ Y and each neighborhood V of q there is {V₁, V₂} in U* such that q ∈ V₁ ⊂ V and Y - V₂ is a neighborhood of q [2]. Let F be a family of functions from a topological space X to semiuniform space (Y, U*). Then F is semiequicontinuous if for each V ∈ U* there is an open cover A of X such that A refines f⁻¹(V) for each f ∈ F [2]. One may easily show that a topological space has a semiuniformity if and only if it is regular.

Let X be a nonempty set. A quasi-uniformity for X is a filter U of reflexive subsets of X × X such that if U ∈ U, there is V ∈ U such that V o V ⊂ U [5]. Let G be a collection of maps from a topological space (X, r) into a quasi-uniform space (Y, U) and let x ∈ X. Then F is quasi-equicontinuous at x provided that for each U ∈ U there exists a neighbor-
hood $N$ of $x$ such that for $f \in F$, $f(N) \subset U(f(x))$ and $F$ is quasi-equicontinuous provided $F$ is quasi-equicontinuous at each $x \in X$. If $y \in Y$ and $U_1 \in \mathcal{U}$ such that $U_1(y)$ is open and $U_2 \in \mathcal{U}$ such that $U_2 \circ U_2 \circ U_2 \circ U_2(y) \subset U_1(y)$ and $U_2 = U_2^{-1}$, then $I = \{U_1(y), \bigcup \text{int} U_2(p); p \notin U_2 \circ U_2(y)\}$ is a two element quasi-uniform cover of $X$. A quasi-uniform space $(X, \mathcal{U})$ is $R_3$, if, given $x \in X$ and $U \in \mathcal{U}$, there exists a symmetric $W \in \mathcal{U}$ such that $W \circ W(x) \subset U(x)$ [3]. It is shown that if $(X, r)$ is regular, then the Pervin quasi-uniformity on $X$ is $R_3$ [5, Theorem 3.17].

2. Topological groups of homeomorphisms.

Theorem 1. Let $(Y, \mathcal{U})$ be an $R_3$-quasi-uniform space. Then the collection of all two element quasi-uniform covers of $Y$ is a semiuniformity for $Y$.

Proof. Let $q \in Y$ and let $V$ be a neighborhood of $q$. Let $U_1 \in \mathcal{U}$ such that $U_1(q) \subset V$ and $U_1(q)$ is open. By hypothesis there is a symmetric entourage $U_2 \in \mathcal{U}$ such that $U_2 \circ U_2 \circ U_2 \circ U_2(q) \subset U_1(q)$. Let $C = \{U_1(q), \bigcup \text{int} U_2(y); y \notin U_2 \circ U_2(q)\}$. Suppose that $x \in Y$ and $x \notin U_1(q)$. Note that if $z \in Y$ and $z \notin U_2 \circ U_2(q)$, then $U_2(z) \subset U_1(q)$. Thus $x \notin U_2 \circ U_2(q)$ and $x \in \text{int} U_2(x)$. Therefore $C$ is an open cover of $Y$. Furthermore, let $p \in U_2(q)$ and suppose that $p \in V_2 = \bigcup \text{int} U_2(y); y \notin U_2 \circ U_2(q)$. Then there exists a $y \in Y$ such that $p \in \text{int} U_2(y)$ and $y \notin U_2 \circ U_2(q)$. But $y \in U_2(p) \subset U_2 \circ U_2(q)$—a contradiction. Then $\mathcal{U}^* = \{C; q \in Y$ and $V$ is a neighborhood of $q\}$ is a semiuniformity for $Y$.

The semiuniformity $\mathcal{U}^*$ of the preceding theorem will be called a quasi-uniform semiuniformity.

Theorem 2. Let $(Y, \mathcal{U})$ be an $R_3$-quasi-uniform space and let $F$ be a family of quasi-equicontinuous functions from a topological space $(X, r)$ into $(Y, \mathcal{U})$. Then $F$ is semiequicontinuous with respect to the quasi-uniform semiuniformity of $\mathcal{U}$.

Proof. Let $\mathcal{U}^*$ be the quasi-uniform semiuniformity of $\mathcal{U}$, let $y, q \in Y$ and $U_1, U_2 \in \mathcal{U}$. Let $I \in \mathcal{U}^*$ such that $I = \{U_1(q), \bigcup \text{int} U_2(y); y \notin U_2 \circ U_2(q)\}$. By hypothesis, for each $x \in X$ there exists a neighborhood $N_x$ of $x$ such that for all $f \in F$, $f(N_x) \subset U_2(f(x))$. It may be seen that $U_2(f(x))$ is contained in either $U_1(q)$ or $V_2 = \bigcup \text{int} U_2(y); y \notin U_2 \circ U_2(q)$ as follows: Let $z_1, z_2 \in U_2(f(x))$, so that $z_1 \notin U_1(q)$ and $z_2 \notin V_2$. Now if $z_2 \notin V_2$, then $z_2 \notin U_2 \circ U_2(q)$ and $(q, z_2) \in U_2 \circ U_2$. Since $(z_2, f(x)) \in U_2$ and $(f(x), z_1) \in U_2$, $z_1 \in U_2 \circ U_2 \circ U_2 \circ U_2(q) \subset U_1(q)$—a contradiction. Thus $\{N_x; x \in X\}$ is the desired open cover of $X$.

The proof of the following theorem is based on the proof of [2, Theorem 4].

Theorem 3. Let $F$ be a family of one-to-one functions of a topological space $(X, r)$ onto itself. Let $\mathcal{U}$ be an $R_3$-quasi-uniformity on $X$ such that
Proof. Throughout the proof, if \( p \in X \) and \( U \in \tau \), then \( W(p, U) \) denotes \( \{ f \in F : f(p) \in U \} \). Let \( \mathcal{U}^* \) be the quasi-uniform semuniformity of \( \mathcal{U} \). Let \( g \in F \), \( p \in X \) and \( V \in \tau \) such that \( W(p, V) \) is a neighborhood of \( g^{-1}(p) \). Since \( \tau \subseteq \tau_0 \) there is \( \{ V_1, V_2 \} \in \mathcal{U}^* \) such that \( g^{-1}(p) \subseteq V_1 \subseteq V \) and \( X - V_2 \) is a \( \tau_0 \) neighborhood of \( g^{-1}(p) \). By Theorem 2, \( F^{-1} \) is semi-equicontinuous with respect to \( \mathcal{U}^* \). Let \( U \) be a \( \tau \)-open cover of \( X \) such that \( \mathcal{U} \) refines \( \{ f(V_1), f(V_2) \} \) for all \( f \in F \) and let \( U \) be a member of \( \mathcal{U} \) that contains \( p \). Then \( W(g^{-1}(p), U) \) is a neighborhood of \( g \). Let \( f \in F \) such that \( f \in W(g^{-1}(p), U) \). Then \( f(g^{-1}(p)) \in U \) and since \( f(g^{-1}(p)) \notin f(V_2) \), \( U \notin f(V_2) \). Hence \( U \subseteq f(V_1) \) and \( f^{-1}(U) \subseteq V_1 \subseteq V \). Consequently, \( f^{-1}(U) \in \mathcal{V} \).

Proposition 4. Let \( (X, \tau) \) be a topological space and let \( F \) be a collection of quasi-equicontinuous functions from \( (X, \tau) \) into a quasi-uniform space \( (Y, \mathcal{U}) \). Then the topology of pointwise convergence on \( F \) is jointly continuous.

Proof. Let \( f \in F \) and let \( x \in X \). For any \( U \in \mathcal{U} \), \( U(f(x)) \) is a neighborhood of \( f(x) \). Let \( V \in \mathcal{U} \) such that \( V \circ V \subseteq U \). By hypothesis there exists a neighborhood \( N \) of \( x \) such that for all \( f \in F \), \( f(N) \subseteq V(f(x)) \). Consider the neighborhoods \( W(x, V)(f) \) and \( N \) of \( f \) and \( x \) respectively. Let \( z \in N \) and let \( g \in W(x, V)(f) \). Then \( (f(x), g(x)), (g(x), g(z)) \in V \) and \( g(z) \in V \circ V(f(x)) \subseteq U(f(x)) \).

Theorem 5 [2, Theorem 5]. Let \( F \) be a semigroup (under composition) of continuous functions from a topological space \( X \) into itself. If the topology of pointwise convergence on \( F \) is jointly continuous, then composition is continuous relative to the topology of pointwise convergence.

Theorem 6. Let \( (X, \tau) \) be any topological space and let \( G \) be a group of homeomorphisms of \( X \) onto \( X \). Let \( \mathcal{U} \) be any \( R_3 \)-quasi-uniformity on \( X \) such that \( \tau \subseteq \tau_0 \) and \( G \) is quasi-equicontinuous with respect to \( \mathcal{U} \). Then \( G \) is a topological group under the topology of pointwise convergence.

Proof. By Proposition 4, the topology of pointwise convergence on \( G \) is jointly continuous. Thus by Theorems 3 and 5, \( G \) is a topological group under the topology of pointwise convergence.

We conclude by giving an example of a non-\( R_0 \) topological space \( (X, \tau) \) with an \( R_3 \)-quasi-uniformity \( \mathcal{U} \) on \( X \) such that \( \tau \subseteq \tau_0 \).

Definition [4]. A preorder on a set \( X \) is any reflexive and transitive relation on \( X \).

Example. Let \( \mathbb{N} \) denote the set of natural numbers. Let \( \leq \) be an antisymmetric preorder on \( \mathbb{N} \) defined as follows:
(i) $x \leq x$ for all $x \in N$,
(ii) $2 \leq 2^k$, $k = 1, 2, 3, \ldots$, and
(iii) $3 \leq 3^k$, $k = 1, 2, 3, \ldots$.

Let $\tau$ be the left topology associated with the preordering $\leq$ [4]. It is not difficult to see that $(N, \tau)$ is a $T_0$ space which is not $T_1$ and hence not $R_0$ [5, Corollary 3.9]. Let $U_n = \{(x, y) | x = y \text{ or } x > n\}$, $\beta = \{U_n | n \in N\}$ and $\mathcal{Q}$ denote the quasi-uniformity on $N$ generated by the base $\beta$ [1]. Then $\mathcal{Q}$ is an $R_3$-quasi-uniformity on $N$ with the property that $\tau$ is properly contained in $\mathcal{Q}$.

REFERENCES


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