A SUFFICIENT CONDITION FOR EVENTUAL DISCONJUGACY

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ABSTRACT. It is known that the scalar equation \( y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \quad t > 0, \quad n > 1, \) is eventually disconjugate if \( p_1, \ldots, p_n \in C[0, \infty) \) and \( \int_0^\infty |p_i(t)|^{i-1} dt < \infty, \quad 1 \leq i \leq n. \) This paper presents a weaker integral condition which also implies that the given equation is eventually disconjugate.

A linear differential equation

(1) \( y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \quad t > 0, \quad (n > 1), \)

is eventually disconjugate if there is an interval \( [a, \infty) \) on which none of its nontrivial solutions has more than \( n-1 \) zeros, counting multiplicities. From a theorem of Willett [4, Theorem 1.4], (1) is eventually disconjugate if \( p_1, \ldots, p_n \in C[0, \infty) \) and

(2) \( \int_0^\infty |p_i(t)|^{i-1} dt < \infty, \quad 1 \leq i \leq n. \)

This paper presents a weaker sufficient condition for eventual disconjugacy.

Let \( I \) be the set of functions defined for large \( t \) and integrable at \( \infty \), let \( A_0 \) be the set of functions in \( I \) and absolutely integrable at \( \infty \), and let \( \Phi \) be the set of functions which are positive, nondecreasing and absolutely continuous for large \( t \). From Abel’s theorem [1, p. 476], \( f/\Phi \in I \) if \( f \in I \) and \( \phi \in \Phi \), and

(3) \( \int_t^\infty f(s)(\phi(s))^{-1} ds = o(1/\phi(s)). \)

(In this paper, \"O\" and \"o\" refer to behavior as \( t \to \infty.\))

For \( i \geq 1 \), define \( A_i \) as follows: \( f \in A_i \) if and only if \( f \in I \) and there are functions \( \phi_1, \ldots, \phi_i \) in \( \Phi \) such that if

(4) \( f_0 = f, \)

(5) \( Q_j(t) = \int_t^\infty f_{j-1}(s)(\phi_j(s))^{-1} ds, \quad 1 \leq j \leq i, \)

(6) \( f_j(t) = t^{-1}\phi_j(t)Q_j(t), \quad 1 \leq j \leq i, \)

and

(7) \( g_j(t) = \phi_j'(t)Q_j(t), \quad 1 \leq j \leq i, \)

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then \( f_0', \ldots, f_{i-1} \in \mathcal{I} \) and \( \ell, g_1, \ldots, g_i \in A_{\mathcal{I}} \). Finally, define

\[
B_{k-1} = \bigcup_{j=0}^{k-1} A_j, \quad k \geq 1.
\]

Our main result is

**Theorem 1.** If \( p_1, \ldots, p_n \in C[0, \infty) \) and

\[
p_{k-1} \in B_{k-1}, \quad 1 \leq k \leq n,
\]

then (1) is eventually disconjugate.

To prove this theorem, we will show that its hypothesis implies that (1) has a fundamental set of solutions \( y_0, \ldots, y_{n-1} \) such that

\[
y^{(j)}(t) = \begin{cases} 
  t^{-j}(1 + o(1))/(r - j)! & 0 \leq j < r, \\
  o(t^{-j}), & r + 1 \leq j \leq n - 1.
\end{cases}
\]

If (10) holds, then the Wronskians \( W_r = W(y_0, \ldots, y_r) \) satisfy

\[
W_r(t) = 1 + o(1), \quad 0 \leq r \leq n - 1,
\]

and are therefore positive on some interval \([a, \infty)\). Because of this, \( \text{Pólya's disconjugacy condition} \) [1] implies that (1) is disconjugate on \([a, \infty)\).

(To see that (10) implies (11), observe that a typical term in the expansion of \( W_r(t) \) according to the definition of determinant is of the form

\[
\pm \Pi_{i=0}^{r-1} y_i^{(j_i)}(t), \quad \text{where } j_0, \ldots, j_{r-1} \text{ is a permutation of } 0, \ldots, r - 1.\]

The product for which \( j_i = i \) \((0 \leq i \leq r - 1)\) equals \( 1 + o(1) \), from (10). Every other product is of the order \( \Pi_{i=0}^{r-1} \mathcal{O}(t^{r-j_i}) \), where \( \mathcal{O} \) can be replaced by \( \mathcal{O} \) in at least one factor. Since \( \sum_{i=0}^{r-1} (i - j_i) = 0 \), every such product equals \( o(1) \)).

We will use the contraction mapping principle to show that \( y_0, \ldots, y_{n-1} \) exist. The subspace \( P_{r, \infty} \) of \( C(n-1)[0, \infty) \) consisting of functions such that \( y^{(j)}(t) = \mathcal{O}(t^{-j}) \), \( 0 \leq j \leq n - 1 \), is a Banach space under the norm

\[
\sigma_r(t; y) = \sum_{j=0}^{n-1} \sup_{s \geq t} |s^{j-r} y^{(j)}(s)|.
\]

With

\[
(\text{My})(t) = \sum_{k=1}^{n} p_k(t) y^{(n-k)}(t)
\]

(cf. (1)), define the mappings \( T_0, \ldots, T_{n-1} \) by

\[
(T_r y)(t) = 1 + \int_{t}^{\infty} \frac{(t - s)^{n-1}}{(r-j)!} (\text{My})(s) \, ds
\]
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\[ (T_r y)(t) = \int_{t_0}^t \lambda (\lambda - \rho)^{n-1} (My)(s) ds, \]

\[ r = 1, \ldots, n - 1. \]

We will show that (9) implies \( T_r \) is a contraction mapping of \( P_{r|t_0, \infty} \) into itself if \( t_0 \) is sufficiently large. It will then follow from the contraction mapping principle [2, p. 11] that there is a function \( y_r \) in \( P_{r|t_0, \infty} \) such that \( T_r y_r = y_r \). It is then straightforward to verify that \( y_r \) satisfies (1) and (10).

Thus, the proof of Theorem 1 is reduced to showing that \( T_r \) has the stated property. We do this by means of the following lemmas.

Lemma 1. Suppose \( f \in A_t, h \in C^i([t_0, \infty)) (t_0 \geq 0), \) and

\[ h^{(j)}(t) = O(t^{-i}), \quad 0 \leq j \leq i. \]

Then the integral

\[ \int_t^\infty s^{-\alpha} f(s) h(s) ds, \quad \alpha \geq 0, \quad t \geq t_0, \]

exists. Moreover, there is a function \( m_i \), which depends on \( f \) but not on \( h \) or \( t_0 \), such that

\[ m_i(t) = o(1) \]

and

\[ \left| \int_t^\infty s^{-\alpha} f(s) h(s) ds \right| \leq m_i(t) \sup_{s \geq t} |s^i h^{(i)}(s)|, \]

\[ 0 < \alpha < n - 1, \quad t \geq t_0. \]

Proof. If \( i = 0 \), (16) converges absolutely and (17) and (18) hold, with \( m_0(t) = \int_t^\infty |f(s)| ds \). If \( i \geq 1 \), define

\[ h_0 = h, \quad h_r = \frac{1}{r} h_{r-1} - \alpha h_{r-1}, \quad 1 \leq r \leq i, \]

and observe that \( h_0, \ldots, h_i \) are bounded because of (15).

Repeated integration by parts yields

\[ \int_t^{t_1} s^{-\alpha} f(s) h(s) ds = - \sum_{j=1}^i Q_j(s) \phi_j(s) s^{-\alpha} h_{j-1}(s) \bigg|_t^{t_1} \]

\[ + \sum_{j=1}^i \int_t^{t_1} s^{-\alpha} g_j(s) h_{j-1}(s) ds \]

\[ = \int_t^{t_1} \sum_{j=1}^i g_j(s) h_{j-1}(s) ds, \quad t_1 \geq t \geq t_0. \]
where \( \phi_j, f_j, g_j, Q_j \) (\( 1 \leq j \leq i \)) are the functions introduced in defining the class \( A_i \) (cf. (4)–(7)) and \( b_0, \ldots, b_i \) are as defined in (19). Since \( \lim_{t \to -\infty} Q_j(t) \phi_j(t) = 0 \), \( 1 \leq j \leq i \), (by Abel's lemma; cf. (3)), the boundedness of \( b_0, \ldots, b_i \) and the absolute integrability of \( f_i \) and \( g_1, \ldots, g_i \) at \( \infty \) imply that the right side of (20) converges as \( t_1 \to \infty \). Hence, the integral (16) converges, and it is easy to verify from (20) that (17) and (18) hold, with

\[
m_i(t) = K_i \left[ \int_t^\infty |f_i(s)| \, ds + \sum_{j=1}^i \left( |Q_j(t)| \phi_j(t) + \int_t^\infty |g_j(s)| \, ds \right) \right],
\]

where \( K_i \) is a constant, which does not depend on \( h \), such that

\[
\sup_{s \geq t} |h_j(s)| \leq K_i \sum_{j=0}^i \sup_{s \geq t} |s^j h^{(j)}(s)|, \quad 0 \leq r \leq i, \quad 0 \leq \alpha \leq n - 1.
\]

The existence of \( K_i \) follows from (15) and (19).

The next lemma follows immediately from (8) and Lemma 1.

**Lemma 2.** The integral (16) exists if \( f \in B_{k-1}, x \in C^{(k-1)}[t_0, \infty) \), and

\[
x^{(j)}(t) = O(t^{-j}), \quad 0 \leq j \leq k - 1.
\]

Moreover, there is a function \( \mu_{k-1} \), which depends on \( f \), but not on \( h \) or \( t_0 \), such that

\[
\mu_{k-1}(t) = o(1)
\]

and

\[
\left| t^\alpha \int_t^\infty s^{-\alpha} f(s) h(s) \, ds \right| \leq \mu_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \geq t} |s^j h^{(j)}(s)|,
\]

\[
0 \leq \alpha \leq n - 1, \quad t \geq t_0.
\]

**Lemma 3.** If the hypotheses of Theorem 1 are satisfied, then the function \( z \), defined by

\[
z_r(t) = \int_t^\infty \frac{(t-s)^{n-r-1}}{(n-r-1)!} (M y)(s) \, ds, \quad 0 \leq r \leq n - 1,
\]

(cf. (12)) is in \( C^{(n-r)}[t_0, \infty) \) if \( y \in P_r[t_0, \infty) \). Furthermore,

\[
t^i |z_r^{(i)}(t)| \leq G_r(t) \sigma_r(t; y), \quad 0 \leq i \leq n - r - 1, \quad t \geq t_0,
\]

where

\[
G_r(t) = o(1)
\]

and \( G_r \) depends on the operator \( M \), but not on \( y \) or \( t_0 \).

**Proof.** Integrals of the form...
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(26) \[ \int_s^\infty s^q p_k(s) y^{(n-k)}(s) ds, \quad 0 \leq q \leq n - r - 1, \]

which appear in (23), can be rewritten as

\[ \int_s^\infty s^{-n + q + r + 1} f_k(s) x_k(s) ds, \quad \text{with } f_k(t) = t^{k-1} p_k(t) \]

and

(27) \[ x_k(t) = t^{n-k-r}(n-k)(t). \]

Since \( f_k \in B_{k-1} \) and \( x_k \) satisfies (21), Lemma 2 implies that (26) exists and, from (22),

(28) \[ \left| \int_s^\infty s^{-n - r - 1} f_k(s) y^{(n-k)}(s) ds \right| \leq \lambda_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \geq t} |s^j x_k^{(j)}(s)|, \]

where \( \lambda_{k-1} \) is as defined in Lemma 2, with \( f = f_k \). From (27), there is a constant \( \lambda_{r_k} \) such that

(29) \[ \sum_{j=0}^{k-1} \sup_{s \geq t} |s^j x_k^{(j)}(s)| \leq \frac{\lambda_{r_k}}{2} \sigma_r(t; y) \]

for every \( y \) in \( P_{t_0, \infty} \), and now (24) and (25) follow from (23), (28), (29) and the inequality

\[ \frac{1}{(n-r-i-1)!} \sum_{\nu=0}^{n-r-i-1} \binom{n-r-i-1}{\nu} = \frac{2^{n-r-i-1}}{(n-r-i-1)!} \leq 2, \]

if we take

\[ G_r(t) = \sum_{k=1}^{n} \lambda_{r_k} \mu_{k-1}(t). \]

Lemma 3 implies that \( T_r \), as defined by (13) and (14), maps \( P_{t_0, \infty} \) into itself, for any \( t_0 \geq 0 \). Moreover, if \( y \) and \( \tilde{y} \) are both in \( P_{t_0, \infty} \), routine estimates based on (13), (14) and (24) yield

(30) \[ \sigma_r(t_0; T_r y - T_r \tilde{y}) \leq n \tilde{G}_r(t_0) \sigma_r(t_0; y - \tilde{y}), \]

where

\[ \tilde{G}_r(t_0) = \sup_{t \geq t_0} G_r(t). \]

Because of (25), we can choose \( t_0 \) so that \( \tilde{G}_r(t_0) < 1/n \), and then (30) implies that \( T_r \) is a contraction mapping of \( P_{t_0, \infty} \) into itself.

The fixed point (function) \( y_r \) of \( T_r \) satisfies (1) on \( (t_0, \infty) \), and can be extended as a solution of (1) over \( (0, \infty) \). To see that \( y_r \) satisfies (10) for
\[ r \leq j \leq n - 1, \] observe that
\[ y_r^{(j)}(t) = \delta_{rij} + \int_0^t \frac{(t-s)^{n-j-1}}{(n-j-1)!} (My)^r(s)ds, \quad r \leq j \leq n - 1, \]
and apply Lemma 3, with \( y = y_r \). For \( 0 \leq j \leq r - 1, \)
\[ y_r^{(j)}(t) = \frac{r-j}{(r-j)!} + \int_0^t \frac{(t-\lambda)^{r-j-1}}{(r-j-1)!} z_r(\lambda) d\lambda, \]
with \( z_r \) as defined by (23), with \( y = y_r \). However,
\[ \left| t^{-r} \int_0^t (t-\lambda)^{r-j-1} y_r(\lambda) d\lambda \right| \leq t^{-1} \int_0^t |z_r(\lambda)| d\lambda, \]
which approaches zero as \( t \to \infty \). (This is obvious if the last integral on
the right converges as \( t \to \infty \), and it follows from L'Hospital's rule and Lem-
ma 3 if it diverges.) Thus, \( y_r \) satisfies (10) for \( 0 \leq j \leq r - 1, \) and the proof of
Theorem 1 is complete.

To the author's knowledge, the classes \( A_1, A_2, \ldots \) have not been pre-
viously considered in the literature, and virtually all questions about them
are open. For example, is \( A_{i-1} \subset A_i \)? How can one construct functions which
are in \( A_i \), but not in \( B_{i-1} \)? Since we cannot answer these questions yet, we
will for the present simply exhibit classes of functions which are in \( A_1 \) and
\( A_2 \), but not in \( A_0 \). This will show that Theorem 1 implies eventual discon-
jugacy for some equations which do not satisfy (2).

**Theorem 2.** Suppose \( F \) is continuous and has a bounded antiderivative
\( F \) on \([0, \infty)\), and \( \psi \) is absolutely continuous and approaches zero mono-
tonically as \( t \to \infty \). Then the function \( f = F \psi \) is in \( A^1 \) if
\[ \int_0^\infty t^{-1} |\psi(t)| dt < \infty. \]  

**Proof.** Take \( \phi_1 = 1 \) in (5). Then
\[ Q_1(t) = \int_0^t F(s)\psi(s) ds = -F_1(t)\psi(t) - \int_0^t F_1(s)\psi'(s) ds, \]
and our hypotheses imply that \( Q_1(t) = O(\psi(t)) \), and therefore, from (6), \( f_1(t) =
O(t^{-1}\psi(t)) \), so that (31) implies \( f_1 \in A_0 \). Since \( g_1 = 0 \), it follows that \( f \in A_1 \).

As an application of Theorem 2, consider the iterated logarithms:
\[ L_0(t) = t, \quad L_1(t) = \log t, \quad L_2(t) = \log \log t, \ldots, \quad L_i(t) = \log L_{i-1}(t). \]
Since \( L_i'(t) = [\prod_{j=0}^{i-1} L_j(t)]^{-1}, \) \( i \geq 1, \) the function
\[ \psi(t) = [L_i(t + \rho)]^{-1} \prod_{j=0}^{i-1} L_j(t + \rho)^{-1}, \quad i \geq 1, \]
where $p$ is a positive constant such that $\psi$ is defined on $[0, \infty)$) has the properties required in Theorem 2, provided $\alpha > 1$. Theorems 1 and 2 imply, for example, that the equation

$$y^{(n)} + \frac{t^{-n+1} \left[ \log \left( \frac{t}{(t+2e)} \right) \right]^{-3/2} \sin t}{\log (t + 2e)} y = 0$$

is eventually disconjugate if $n \geq 2$. This equation does not satisfy (2).

**Theorem 3.** Suppose $F$ is continuous on $[0, \infty)$ and has a bounded antiderivative $F'$, which in turn has a bounded antiderivative $F''$. Suppose also that $\psi$ and $\psi'$ both tend monotonically to zero as $t \to \infty$, and $\psi'$ is absolutely continuous. Then the function $f = F\psi$ is in $A_2$.

**Proof.** From (5) (with $\phi_1 = 1$),

$$Q_1(t) = \int_0^t F(s)\psi(s) ds = -F_1(t)\psi(t) + F_2(t)\psi'(t) + \int_t^\infty F_2(s)\psi''(s) ds. \quad (33)$$

Our hypotheses imply that the integral on the right equals $O(\psi'(t))$, so (6) and (33) yield

$$f_1(t) = -F_1(t)t^{-1}\psi(t) + A(t)t^{-1}\psi'(t),$$

where $A$ is continuous and bounded on $[0, \infty)$. Letting $\phi_2 = 1$ in (5) yields

$$Q_2(t) = -\int_t^\infty F_1(s)s^{-1}\psi'(s) ds + \int_t^\infty A(s)s^{-1}\psi'(s) ds \quad (34)$$

$$= F_2(t)t^{-1}\psi(t) + \int_t^\infty F_2(s)(s^{-1}\psi(s))' ds + \int_t^\infty A(s)s^{-1}\psi'(s) ds.$$

Our hypotheses and the boundedness of $A$ imply that each term on the right is $O(t^{-1}\psi(t))$; hence (6) and (34) yield

$$f_2(t) = O(t^{-2}\psi(t)) = o(t^{-2}),$$

and therefore $f_2 \in A_0$. Since $g_1 = g_2 = 0$, it follows that $f \in A_2$.

Theorems 1 and 3 imply, for example, that the equation

$$y^{(n)} + \frac{t^{-n+1} \sin t}{L_k(t + p)} y = 0$$

is eventually disconjugate if $n \geq 3$ and $L_k$ is any iterated logarithm (32) defined on $[p, \infty)$. This equation does not satisfy (2).

**REFERENCES**


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