A SUFFICIENT CONDITION FOR EVENTUAL DISCONJUGACY

WILLIAM F. TRENCH

ABSTRACT. It is known that the scalar equation \( y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \ t > 0, \ n > 1, \) is eventually disconjugate if \( p_1, \ldots, p_n \in C[0, \infty) \) and \( \int_0^\infty |p_i(t)|^{i-1} \ dt < \infty, \ 1 \leq i \leq n. \) This paper presents a weaker integral condition which also implies that the given equation is eventually disconjugate.

A linear differential equation
\[
(1) \quad y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \quad t > 0 \ (n > 1),
\]
is eventually disconjugate if there is an interval \([a, \infty)\) on which none of its nontrivial solutions has more than \(n-1\) zeros, counting multiplicities. From a theorem of Willett \([4, \text{Theorem } 1.4]\), \((1)\) is eventually disconjugate if \( p_1, \ldots, p_n \in C[0, \infty) \) and
\[
(2) \quad \int_0^\infty |p_i(t)|^{i-1} \ dt < \infty, \quad 1 \leq i \leq n.
\]
This paper presents a weaker sufficient condition for eventual disconjugacy.

Let \( l \) be the set of functions defined for large \( t \) and integrable at \( \infty \), let \( A_0 \) be the set of functions in \( l \) and absolutely integrable at \( \infty \), and let \( \Phi \) be the set of functions which are positive, nondecreasing and absolutely continuous for large \( t \). From Abel's theorem \([1, \text{p. } 476]\), \( f/\Phi \in l \) if \( f \in l \) and \( \phi \in \Phi \), and
\[
(3) \quad \int_t^\infty f(s)(\phi(s))^{-1} \ ds = o(1/\phi(s)).
\]
(In this paper, "\( 0^+ \)" and "\( o^+ \)" refer to behavior as \( t \to \infty \).)

For \( i \geq 1 \), define \( A_i \) as follows: \( f \in A_i \) if and only if \( f \in l \) and there are functions \( \phi_1, \ldots, \phi_i \) in \( \Phi \) such that if
\[
(4) \quad f_0 = f,
\]
\[
(5) \quad Q_j(t) = \int_t^\infty f_{j-1}(s)(\phi_j(s))^{-1} \ ds, \quad 1 \leq j \leq i,
\]
\[
(6) \quad f_j(t) = t^{-1}\phi_j(t)Q_j(t), \quad 1 \leq j \leq i,
\]
and
\[
(7) \quad g_j(t) = \phi_j'(t)Q_j(t), \quad 1 \leq j \leq i,
\]
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Our main result is

**Theorem 1.** If \( p_1, \ldots, p_n \in C[0, \infty) \) and

\[
(9) \quad t^{k-1} p_k \in B_{k-1}, \quad 1 \leq k \leq n,
\]

then (1) is eventually disconjugate.

To prove this theorem, we will show that its hypothesis implies that (1) has a fundamental set of solutions \( y_0, \ldots, y_{n-1} \) such that

\[
(10) \quad y_{(j)}(t) = \begin{cases} 
  t^r \left( 1 + o(1) \right) / (r-j)!, & 0 \leq j < r, \\
  o(t^r), & r + 1 \leq j \leq n - 1.
\end{cases}
\]

If (10) holds, then the Wronskians \( W_r = W(y_0, \ldots, y_r) \) satisfy

\[
(11) \quad W_r(t) = 1 + o(1), \quad 0 \leq r \leq n - 1,
\]

and are therefore positive on some interval \([a, \infty)\). Because of this, Pólya's disconjugacy condition \([5]\) implies that (1) is disconjugate on \([a, \infty)\).

To see that (10) implies (11), observe that a typical term in the expansion of \( W_r(t) \) according to the definition of determinant is of the form

\[
\pm \prod_{i=0}^{r-1} y_{(i)}(t), \quad \text{where} \quad \{j_0, \ldots, j_{r-1}\} \text{ is a permutation of} \{0, \ldots, r-1\}.
\]

The product for which \( j_i = i \) \((0 \leq i \leq r-1)\) equals \( 1 + o(1) \), from (10). Every other product is of the order \( \prod_{i=0}^{r-1} O(t^{i-j_i}) \), where "\( O \)" can be replaced by "\( o \)" in at least one factor. Since \( \prod_{i=0}^{r-1} (i-j_i) = 0 \), every such product equals \( o(1) \).

We will use the contraction mapping principle to show that \( y_0, \ldots, y_{n-1} \) exist. The subspace \( P_r[0, \infty) \) of \( C^{n-1}[0, \infty) \) consisting of functions such that \( y^{(j)}(t) = O(t^{r-j}), 0 \leq j \leq n - 1 \), is a Banach space under the norm

\[
\sigma_r(t; y) = \sum_{j=0}^{n-1} \sup_{s \geq t} |s^{j-r} y_{(j)}(s)|.
\]

With

\[
(12) \quad (My)(t) = \sum_{k=1}^{n} p_k(t) y^{(n-k)}(t)
\]

(cf. (1)), define the mappings \( T_0, \ldots, T_{n-1} \) by

\[
(13) \quad (T_r y)(t) = 1 + \int_{0}^{\infty} (t-s)^{n-1} \frac{(My)(s) ds}{(n-r)!}
\]
and
\[
(T_y)(t) = \frac{t}{r!} + \int_{t_0}^{t} \frac{(t - \lambda)^{r-1}}{(r - 1)!} d\lambda \int_{\lambda}^{\infty} \frac{(\lambda - s)^{n-r-1}}{(n - r - 1)!} (My)(s) \, ds,
\]
(14)
\[
r = 1, \ldots, n - 1.
\]

We will show that (9) implies \( T_r \) is a contraction mapping of \( P_{[t_0, \infty)} \) into itself if \( t_0 \) is sufficiently large. It will then follow from the contraction mapping principle [2, p. 11] that there is a function \( y_r \) in \( P_{[t_0, \infty)} \) such that \( T_r y_r = y_r \). It is then straightforward to verify that \( y_r \) satisfies (1) and (10).

Thus, the proof of Theorem 1 is reduced to showing that \( T_r \) has the stated property. We do this by means of the following lemmas.

**Lemma 1.** Suppose \( f \in A_t \), \( h \in C^{(i)}([t_0, \infty)) \) \( (t_0 \geq 0) \), and
\[
h^{(j)}(t) = O(t^{-j}), \quad 0 \leq j \leq i.
\]
(15)

Then the integral
\[
\int_{t}^{\infty} s^{-\alpha f(s)} h(s) \, ds, \quad \alpha \geq 0, \quad t \geq t_0,
\]
exists. Moreover, there is a function \( m_i \), which depends on \( f \) but not on \( h \) or \( t_0 \), such that
\[
m_i(t) = o(1)
\]
(17)

and
\[
\left| t^{\alpha} \int_{t}^{\infty} s^{-\alpha f(s)} h(s) \, ds \right| \leq m_i(t) \sum_{j=0}^{i} \sup_{s \geq t} |s^{j} h^{(j)}(s)|,
\]
(18)
\[
0 < \alpha < n - 1, \quad t \geq t_0.
\]

**Proof.** If \( i = 0 \), (16) converges absolutely and (17) and (18) hold, with \( m_0(t) = \int_{t}^{\infty} |f(s)| \, ds \). If \( i \geq 1 \), define
\[
h_0 = h, \quad h_r = th_{r-1} - \alpha h_{r-1}, \quad 1 \leq r \leq i,
\]
(19)

and observe that \( h_0, \ldots, h_i \) are bounded because of (15).

Repeated integration by parts yields
\[
\int_{t}^{t_1} s^{-\alpha f(s)} h(s) \, ds = - \sum_{j=1}^{i} Q_j(s) \phi_j(s) s^{-\alpha h_{j-1}}(s) \bigg|_{t}^{t_1} + \sum_{j=1}^{i} \int_{t}^{t_1} s^{-\alpha g_j(s)} h_{j-1}(s) \, ds,
\]
(20)
\[
t_1 \geq t \geq t_0.
\]
where \( \phi_j, f_j, g_j, Q_j \) (\( 1 \leq j \leq i \)) are the functions introduced in defining the class \( A_i \) (cf. (4)-(7)) and \( h_0, \ldots, h_i \) are as defined in (19). Since
\[
\lim_{t \to \infty} Q_j(t) \phi_j(t) = 0, \quad 1 \leq j \leq i,
\]
(by Abel's lemma; cf. (3)), the boundedness of \( h_0, \ldots, h_i \) and the absolute integrability of \( f_i, g_1, \ldots, g_i \) at \( \infty \) imply that the right side of (20) converges as \( t_1 \to \infty \). Hence, the integral (16) converges, and it is easy to verify from (20) that (17) and (18) hold, with
\[
m_i(t) = K_i \left[ \int_t^\infty |f_i(s)| \, ds + \sum_{j=1}^i \left( |Q_j(t)| \phi_j(t) + \int_t^\infty |g_j(s)| \, ds \right) \right],
\]
where \( K_i \) is a constant, which does not depend on \( h \), such that
\[
\sup_{s \geq t} |h_r(s)| \leq K_i \sum_{j=0}^i \sup_{s \geq t} |s^j h^{(j)}(s)|, \quad 0 \leq r \leq i, \quad 0 \leq \alpha \leq n - 1.
\]
The existence of \( K_i \) follows from (15) and (19).

The next lemma follows immediately from (8) and Lemma 1.

**Lemma 2.** The integral (16) exists if \( f \in B_{k-1} \), \( x \in C^{(k-1)}(t_0, \infty) \), and
\[
x^{(j)}(t) = O(t^{-j}), \quad 0 \leq j \leq k - 1.
\]
Moreover, there is a function \( \mu_{k-1} \), which depends on \( f \), but not on \( h \) or \( t_0 \), such that
\[
\mu_{k-1}(t) = o(1)
\]
and
\[
\left| t^\alpha \int_t^\infty s^{-\alpha} f(s) h(s) \, ds \right| \leq K_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \geq t} |s^j h^{(j)}(s)|,
\]
\[
0 \leq \alpha \leq n - 1, \quad t \geq t_0.
\]

**Lemma 3.** If the hypotheses of Theorem 1 are satisfied, then the function \( z_r \), defined by
\[
z_r(t) = \int_t^\infty (t-s)^{n-r-1} (M(y) s) \, ds, \quad 0 \leq r \leq n - 1,
\]
(cf. (12)) is in \( C^{(n-r)}(t_0, \infty) \) if \( y \in P_r[t_0, \infty) \). Furthermore,
\[
t^i |z_r^{(i)}(t)| \leq G_r(t) \sigma_r(t; y), \quad 0 \leq i \leq n - r - 1, \quad t \geq t_0,
\]
where
\[
G_r(t) = o(1)
\]
and \( G_r \) depends on the operator \( M \), but not on \( y \) or \( t_0 \).

**Proof.** Integrals of the form
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\[ (26) \quad \int_{1}^{\infty} s^{q} p_{k}(s) y^{(n-k)}(s) \, ds, \quad 0 \leq q \leq n - r - 1, \]

which appear in (23), can be rewritten as

\[ \int_{1}^{\infty} s^{-n+q+r+1} f_{k}(s)x_{k}(s) \, ds, \quad \text{with} \quad f_{k}(t) = t^{k-1} p_{k}(t) \]

and

\[ (27) \quad x_{k}(t) = t^{n-k-r} y^{(n-k)}(t). \]

Since \( f_{k} \in B_{k-1} \) and \( x_{k} \) satisfies (21), Lemma 2 implies that (26) exists and, from (22),

\[ (28) \quad \left| t^{n-q-r-1} \int_{1}^{\infty} s^{q} p_{k}(s) y^{(n-k)}(s) \, ds \right| \leq \mu_{k-1}(t) \sum_{j=0}^{k-1} \sup_{s \geq t} |s^{j} x_{k}^{(j)}(s)|, \]

where \( \mu_{k-1} \) is as defined in Lemma 2, with \( f = f_{k} \). From (27), there is a constant \( \lambda_{rk} \) such that

\[ (29) \quad \sum_{j=0}^{k-1} \sup_{s \geq t} |s^{j} x_{k}^{(j)}(s)| \leq \frac{\lambda_{rk}}{2} \sigma_{r}(t; y) \]

for every \( y \) in \( P_{r}[t_{0}, \infty) \), and now (24) and (25) follow from (23), (28), (29) and the inequality

\[ \frac{1}{(n-r-i-1)!} \sum_{\nu=0}^{n-r-i-1} \binom{n-r-i-1}{\nu} = \frac{2^{n-r-i-1}}{(n-r-i-1)!} \leq 2, \]

if we take

\[ G_{r}(t) = \sum_{k=1}^{n} \lambda_{rk} \mu_{k-1}(t). \]

Lemma 3 implies that \( T_{r} \), as defined by (13) and (14), maps \( P_{r}[t_{0}, \infty) \) into itself, for any \( t_{0} \geq 0 \). Moreover, if \( y \) and \( \tilde{y} \) are both in \( P_{r}[t_{0}, \infty) \), routine estimates based on (13), (14) and (24) yield

\[ (30) \quad \sigma_{r}(t_{0}; T_{r} y - T_{r} \tilde{y} \leq n G_{r}(t_{0}) \sigma_{r}(t_{0}; y - \tilde{y}), \]

where

\[ G_{r}(t_{0}) = \sup_{t \geq t_{0}} G_{r}(t). \]

Because of (25), we can choose \( t_{0} \) so that \( G_{r}(t_{0}) < 1/n \), and then (30) implies that \( T_{r} \) is a contraction mapping of \( P_{r}[t_{0}, \infty) \) into itself.

The fixed point (function) \( y_{r} \) of \( T_{r} \) satisfies (1) on \( (t_{0}, \infty) \), and can be extended as a solution of (1) over \( (0, \infty) \). To see that \( y_{r} \) satisfies (10) for
$r \leq j \leq n - 1$, observe that

$$y_r^{(j)}(t) = \delta_{rt} + \int_t^\infty \frac{(t-s)^{n-j-1}}{(n-j-1)!} (My_r)(s)ds, \quad r \leq j \leq n - 1,$$

and apply Lemma 3, with $y = y_r$. For $0 \leq j \leq r - 1$,

$$y_r^{(j)}(t) = \frac{t^{r-j}}{(r-j)!} + \int_0^t \frac{(t-\lambda)^{r-j-1}}{(r-j-1)!} z_r(\lambda) d\lambda,$$

with $z_r$ as defined by (23), with $y = y_r$. However,

$$\int_0^t |z_r(\lambda)| d\lambda \leq t \int_0^t |z_r(\lambda)| d\lambda,$$

which approaches zero as $t \to \infty$. (This is obvious if the last integral on the right converges as $t \to \infty$, and it follows from L'Hôpital's rule and Lemma 3 if it diverges.) Thus, $y_r$ satisfies (10) for $0 \leq j \leq r - 1$, and the proof of Theorem 1 is complete.

To the author's knowledge, the classes $A_1, A_2, \ldots$ have not been previously considered in the literature, and virtually all questions about them are open. For example, is $A_{i-1} \subset A_i$? How can one construct functions which are in $A_i$, but not in $B_{i-1}$? Since we cannot answer these questions yet, we will for the present simply exhibit classes of functions which are in $A_1$ and $A_2$, and not in $A_0$. This will show that Theorem 1 implies eventual disconjugacy for some equations which do not satisfy (2).

**Theorem 2.** Suppose $F$ is continuous and has a bounded antiderivative $F_1$ on $[0, \infty)$, and $\psi$ is absolutely continuous and approaches zero monotonically as $t \to \infty$. Then the function $f = F\psi$ is in $A_1$ if

$$\int_0^\infty t^{-1} |\psi(t)| dt < \infty.$$

**Proof.** Take $\phi_1 = 1$ in (5). Then

$$Q_1(t) = \int_0^t F(s)\psi(s) ds = -F_1(t)\psi(t) - \int_t^\infty F_1(s)\psi'(s) ds,$$

and our hypotheses imply that $Q_1(t) = O(\psi(t))$, and therefore, from (6), $f_1(t) = O(t^{-1}\psi(t))$, so that (31) implies $f_1 \in A_1$. Since $g_1 = 0$, it follows that $f \in A_1$.

As an application of Theorem 2, consider the iterated logarithms:

$$L_0(t) = t, \quad L_1(t) = \log t, \quad L_2(t) = \log (\log t), \ldots, \quad L_i(t) = \log L_{i-1}(t).$$

Since $L_i'(t) = \prod_{j=0}^{i-1} L_j(t)$, $i \geq 1$, the function

$$\psi(t) = [L_i(t + \rho)]^{-1}, \quad i \geq 1,$$
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(where $p$ is a positive constant such that $\psi$ is defined on $[0, \infty)$) has the properties required in Theorem 2, provided $\alpha > 1$. Theorems 1 and 2 imply, for example, that the equation

$$y^{(n)} + \frac{t^{-n+1} \left[ \log \left( \log (t + 2e) \right) \right]^{-3/2} \sin t}{\log (t + 2e)} y = 0$$

is eventually disconjugate if $n \geq 2$. This equation does not satisfy (2).

**Theorem 3.** Suppose $F$ is continuous on $[0, \infty)$ and has a bounded antiderivative $F_1$, which in turn has a bounded antiderivative $F_2$. Suppose also that $\psi$ and $\psi'$ both tend monotonically to zero as $t \rightarrow \infty$, and $\psi'$ is absolutely continuous. Then the function $f = F\psi$ is in $A_2$.

**Proof.** From (5) (with $\phi_1 = 1$),

$$Q_1(t) = \int_0^t F(s)\psi(s) ds = -F_1(t)\psi(t) + F_2(t)\psi'(t) + \int_t^\infty F_2(s)\psi''(s) ds.$$  

Our hypotheses imply that the integral on the right equals $O(\psi'(t))$, so (6) and (33) yield

$$f_1(t) = -F_1(t)t^{-1}\psi(t) + A(t)t^{-1}\psi'(t),$$

where $A$ is continuous and bounded on $[0, \infty)$. Letting $\phi_2 = 1$ in (5) yields

$$Q_2(t) = -\int_t^\infty F_1(s)s^{-1}\psi(s) ds + \int_t^\infty A(s)s^{-1}\psi'(s) ds$$

$$= F_2(t)t^{-1}\psi(t) + \int_t^\infty F_2(s)(s^{-1}\psi(s))' ds + \int_t^\infty A(s)s^{-1}\psi'(s) ds.$$  

Our hypotheses and the boundedness of $A$ imply that each term on the right is $O(t^{-1}\psi(t))$; hence (6) and (34) yield

$$f_2(t) = O(t^{-2}\psi(t)) = o(t^{-2}),$$

and therefore $f_2 \in A_2$. Since $g_1 = g_2 = 0$, it follows that $f \in A_2$.

Theorems 1 and 3 imply, for example, that the equation

$$y^{(n)} + \frac{t^{-n+1} \sin t}{L_k(t + p)} y = 0$$

is eventually disconjugate if $n \geq 3$ and $L_k$ is any iterated logarithm (32) defined on $[p, \infty)$. This equation does not satisfy (2).

**REFERENCES**


DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19104