ON THE RANGE OF A HYPONORMAL DERIVATION

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ABSTRACT. The inner derivation induced by a hyponormal operator has closed range if and only if the operator has finite spectrum.

Let $\mathfrak{B}(H)$ denote the algebra of all bounded linear operators on a Hilbert space $H$. Define the inner derivation

$$\Delta_A : \mathfrak{B}(H) \to \mathfrak{B}(H)$$

by $\Delta_A(X) = AX -XA$ for $A, X \in \mathfrak{B}(H)$. For $T \in \mathfrak{B}(H)$ normal, Anderson and Foias [1] proved that the range of $\Delta_T$ (denoted by $\mathcal{R}(\Delta_T)$) is norm closed if and only if $\sigma(T)$ is finite. Their proof uses a number of deep results on decomposable operators and asymptotic commutativity. In this note we present a simple proof which enables us to extend their result to hyponormal operators.

The method of proof also permits us to answer partially a question raised by S. R. Caradus. To wit, when is $\mathcal{R}(\Delta_T) \cap K = \Delta_T(K)$ where $K$ is the ideal of compact operators. When $T$ is hyponormal we show that equality holds if and only if $\sigma(T)$ is finite. The following result is a slight variation on a well-known result. See [3, Lemma 2] and subsequent material for example.

**Lemma 1.** Let $T \in \mathfrak{B}(H)$ be hyponormal. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct nonisolated boundary points of $\sigma(T)$. Let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive (nonzero) numbers converging to 0. Then there exists an orthonormal sequence $\{f_n\}_{n=1}^\infty$ of vectors from $H$ such that

1. $\| (T - \lambda_n) f_n \| < \epsilon_n$ for $n = 1, 2, \ldots$, and
2. $(f_j, T f_n) = 0$ for $n = 1, \ldots, j - 1$.

**Theorem 1.** Let $T \in \mathfrak{B}(H)$ be hyponormal. Then $\mathcal{R}(\Delta_T)$ is norm closed if and only if $\sigma(T)$ is finite.

**Proof.** Let $\sigma(T)$ be infinite. Then $\sigma(T)$ has an infinite number of boundary points. We distinguish two cases. If $\sigma(T)$ has an infinite number of isolated boundary points $\{\lambda_n\}_{n=1}^\infty$, then by a well-known result [2] there exists an orthonormal sequence $\{f_n\}_{n=1}^\infty$ such that $T f_n = \lambda_n f_n$ (this case is much easier to handle and the reader may wish to work it out first). If $\sigma(T)$ has an infinite

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number of distinct nonisolated boundary points \( \{ \lambda_n \}_{1}^{\infty} \), we can apply the previous lemma. In this case there exists an orthonormal sequence \( \{ f_n \}_{1}^{\infty} \) such that \( \| (T - \lambda_n) f_n \| < \epsilon_n \) and \( \langle f_j, T f_n \rangle = 0 \) for \( j > n \). We may further assume the \( \lambda_n \)'s converge and we choose the \( \epsilon_n \)'s to satisfy the following conditions:

1. \( \epsilon_n > \epsilon_{n+1} > \cdots \);
2. \( \epsilon_n \leq |\lambda_{n+1} - \lambda_n|^2 \) for \( n = 1, 2, \ldots \);
3. \( \sum_{n=1}^{\infty} \epsilon_n \eta_n < \infty \) where \( \eta_n = \max |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \).

We set \( H_1 = \text{clm} \{ f_n \}_{1}^{\infty} \) and \( H_2 = H_1^* \). If we write \( T f_n = \mu_n f_n + \delta_n \) where \( (\delta_n, f_n) = 0 \) then \( |\mu_n - \lambda_n| < \epsilon_n \) and \( \| \delta_n \| < \epsilon_n \) for \( n = 1, 2, \ldots \). We will now define operators \( V_m \) such that \( TV_m - V_m T \) will converge in norm to an operator \( A \in \mathcal{B}(H) \), but \( A \neq TW - WT \) for any \( W \in \mathcal{B}(H) \). We define the unbounded operator \( V \) as follows: \( V f_n = |\lambda_{n+1} - \lambda_n|^{-\frac{1}{2}} f_{n+1} \) for \( n = 1, 2, \ldots \) and \( V g = 0 \) for \( g \in H_2 \). Let \( P_m \) be the projection of \( H \) onto \( \text{clm} \{ f_1, \ldots, f_m \} \) and set \( V_m = VP_m \). We claim that \( TV_m - V_m T \) converges in norm to an operator \( A \in \mathcal{B}(H) \). Note first that

\[
(TV_n - V_n T)f_j = \begin{cases} 
|\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} (\mu_{j+1} - \mu_j) f_{j+1} + |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \delta_{j+1} & \text{for } j \leq n, \\
-V_n \delta_j & \text{for } j > n.
\end{cases}
\]

Thus

\[
[\Delta_T(V_n) - \Delta_T(V_m)]f_j = \begin{cases} 
0 & \text{for } j \leq n \leq m, \\
-|\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} (\mu_{j+1} - \mu_j) f_{j+1} + |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \delta_{j+1} & \text{for } n < j \leq m, \\
(V_m - V_n) \delta_j & \text{for } n \leq m < j.
\end{cases}
\]

Note that \( \| V_n \delta_j \| \leq \| V_n \| \| \delta_j \| \leq n \epsilon_j \leq n_j \epsilon_j \) for all \( n, j \). (The last estimate follows by considering the two cases \( n < j \), \( n > j \) and using the fact that \( (\delta_j, f_m) = 0 \) for \( m = j + 1, j + 2, \ldots \) in the latter.) Let \( h \in H_1 \), and write \( h = \sum_{j=1}^{\infty} a_j f_j \). By a standard argument we see that \( \| [\Delta_T(V_n) - \Delta_T(V_m)] b \| \to 0 \) uniformly in \( h \) as \( n, m \to \infty \), since \( |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} |\mu_{j+1} - \mu_j| \to 0 \) as \( j \to \infty \) and \( \sum \epsilon_j \eta_j < \infty \). We still must consider vectors \( g \in H_2 \). For such a \( g \),

\[
(TV_n - V_n T)g = -V_n T g.
\]

Let \( T g = \sum a_j f_j + w \) where \( w \in H_2 \). Then \( T^* f_j = \bar{\mu}_j f_j + \gamma_j \) where \( (\gamma_j, f_j) = 0 \). Since \( T \) is hyponormal \( \| \gamma_j \| < \epsilon_j \). Thus \( a_j = (T g, f_j) = (g, T^* f_j) = (g, \gamma_j) \) and hence \( |a_j| < \epsilon_j \) if \( g \) is a unit vector. Hence

\[
(TV_n - V_n T)g = \sum_{j=1}^{n} \delta_j |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} f_{j+1}.
\]
Finally

\[ \| (\Delta_T(V_n) - \Delta_T(V_m)g) \| \leq \sum_{j=n}^{m} |a_j| |\lambda_{j+1} - \lambda_j|^{-\frac{1}{2}} \leq \sum_{n}^{m} \epsilon_j \eta_j \]

and the last term tends to zero as \( n, m \to \infty \). Thus \( \{\Delta_T(V_n)\} \) is a Cauchy sequence and hence it converges to an operator \( A \in \mathcal{B}(\mathcal{H}) \). To complete the first half of the proof we must show that \( A \neq TW - WT \). Assume the contrary. Thus \( \langle (TW - WT)f_n, f_{n+1} \rangle = \langle Af_n, f_{n+1} \rangle \) for all \( n \) and hence

\[ \langle \mu_{n+1} - \mu_n \rangle (Wf_n, f_{n+1}) + (Wf_n, \gamma_{n+1}) - (W\delta_n, f_{n+1}) \]

\[ = \langle \lambda_{n+1} - \lambda_n \rangle (Wf_n, f_{n+1}) \]

since \( \{\delta_{n+1}, f_{n+1}\} \) and \( \{V\delta_n, f_{n+1}\} \) are zero. Thus \( \|Wf_n, f_{n+1}\| \geq \frac{1}{2} |\lambda_{n+1} - \lambda_n|^{-\frac{1}{2}} \) for large \( n \) since \( \epsilon_n/|\mu_{n+1} - \mu_n| \to 0 \). This implies that \( W \) is unbounded, contrary to assumption. The other half of the proof will be sketched later.

We now turn to the question of Caradus mentioned in the introduction.

**Theorem 2.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be hyponormal. Then \( \mathcal{R}(\Delta_T) \cap \mathcal{K} = \Delta_T(\mathcal{K}) \) if and only if \( \sigma(T) \) is finite.

**Proof.** Again we prove only half the theorem now. Let \( \sigma(T) \) be infinite. Proceed as in Theorem 1 and select \( \lambda_n, f_n \) and \( \epsilon_n \) as before. This time however we define \( Vf_n = f_{n+1} \) for \( n = 1, 2, \ldots \) and \( Vg = 0 \) for \( g \in \mathcal{H}_2 \).

By estimates similar to those in Theorem 1, it is easy to see that \( B = TV - VT \) is compact, (indeed, the operator \( A \) in Theorem 1 is compact.) Note that \( \langle (TV - VT)f_n, f_{n+1} \rangle = \langle \mu_{n+1} - \mu_n \rangle \), since the other terms are zero. Assume that \( B = TW - WT \) for some noncompact \( W \in \mathcal{B}(\mathcal{H}) \). Then

\[ \langle (TW - WT)f_n, f_{n+1} \rangle = \langle \mu_{n+1} - \mu_n \rangle (Wf_n, f_{n+1}) + (Wf_n, \gamma_{n+1}) - (W\delta_n, f_{n+1}) \]

\[ = \langle Bf_n, f_{n+1} \rangle = \langle \mu_{n+1} - \mu_n \rangle \]

Dividing the last equation by \( \langle \mu_{n+1} - \mu_n \rangle \) and letting \( n \to \infty \) we see that \( \langle Wf_n, f_{n+1} \rangle \to 1 \). Thus \( W \) is not compact and therefore \( B \notin \Delta_T(\mathcal{K}) \).

**Remark.** Let us now assume that \( T \) is hyponormal and \( \sigma(T) \) is finite.

In that case \( T \) must be normal. Thus we write \( T = \Sigma_{j=1}^{n} \lambda_j E_j \) when the \( E_j \)'s are just the spectral projections. For \( V \in \mathcal{B}(\mathcal{H}) \) write \( V \) as a matrix \( [V_{ij}] \) on \( \mathcal{H} = \Sigma_{j=1}^{n} \bigoplus E_j \mathcal{H} \). Then the \( ij \) entry in the matrix representation of \( (TV - VT) \) is just \( \lambda_i - \lambda_j \). This observation should make it clear that \( \mathcal{R}(\Delta_T) \) is closed and moreover that \( \mathcal{R}(\Delta_T) \cap \mathcal{K} = \Delta_T(\mathcal{K}) \) since \( \{(\lambda_i - \lambda_j)V_{ij}\} \) is compact if and only if \( V_{ij} \) is compact for all \( i \neq j \).

**Example.** In the case of an arbitrary operator \( T \in \mathcal{B}(\mathcal{H}) \) we note that...
σ(T) finite does not imply $\mathcal{R}(\Delta_T) \cap K = \Delta_T(K)$. For example let $T = \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ where $Q$ is compact. If $R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ then

$$TR - RT = \begin{bmatrix} QC & QD - AQ \\ 0 & -CQ \end{bmatrix}.$$ 

If we set $A = D = 0$ and $C = I$ then the operator $\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}$ is in $\mathcal{R}(\Delta_T) \cap K$. But if $Q$ is a selfadjoint compact operator with trivial kernel then $\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}$ is clearly not in $\Delta_T(K)$.

REFERENCES


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