THE MAPS $BSp(1) \to BSp(n)$

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ABSTRACT. Let $Sp(n)$ be the symplectic Lie group. Then it is known that given a map $f: BSp(1) \to BSp(1)$, $f^*: H^*(BSp(1), \mathbb{Z}) \to H^*(BSp(1), \mathbb{Z})$ is zero or multiplication by the square of an odd integer.

We generalise the latter part of this result using symplectic $K^*$-theory.

We begin with some notation. Let $T' \subset Sp(n)$ be the standard maximal torus [2] and $BSp(n)$ a classifying space [8].

All cohomology will have integer coefficients.

From [5], we have $H^*(BT') \cong \mathbb{Z}[t_1, \ldots, t_n]$, $\dim t_i = 2$. The inclusion $T' \subset Sp(n)$ induces an injection of $H^*(BSp(n))$ onto the Weyl group invariants in $H^*(BT')$.

Let $T \subset Sp(1)$ be the standard maximal torus. Then we extract the following from [3].

Proposition 1. If $f: BSp(1) \to BSp(n)$ is a map and $f^*: H^*(BSp(n)) \to H^*(BSp(1))$, then there is an extension $\phi^*: H^*(BT') \to H^*(BT)$ of $f^*$.

Thus, if $H^*(BT) \cong \mathbb{Z}[$l$]$, then $\phi^* t_i = m(i)t$ for some integer $m(i)$. In this note we prove the following: In the set $\{m(1), m(2), \ldots, m(n)\}$, each even $m(i)$ occurs an even number of times. For this purpose we compute $f^1$:

$$KU^0(BSp(n)) \to KU^0(BSp(1)),$$

where $KU^*$ is complex $K^*$-theory.

From [4] we find that $KU^0(BT') \cong \mathbb{Z}[s_1, \ldots, s_n]$ where $(1 + s_i)$ is the virtual canonical line bundle over $BS^1$. Put $z_i = 1 + s_i$. $KU^0(BSp(n))$ is isomorphic to the Weyl group invariants in $KU^0(BT')$ [4], and the Weyl group acts by permuting the $z_i$ and inverting: $z_i \to z_i^{-1}$. Hence $KU^0(BSp(n)) \cong \mathbb{Z}[[y_1, \ldots, y_n]]$, $y_i = i$th elementary symmetric function in $(z_i + z_i^{-1} - 2)$.

For $BSp(1)$, put $y_1 = y$.

Let $G$ be a compact connected Lie group and $R(G)$ its complex representation ring. Then in [4, p. 29], an isomorphism $\delta: \hat{R}(G) \to KU^0(BG)$ is described (here $\hat{R}(G)$ is the completion of $R(G)$ under the augmentation topology). There are also monomorphisms $\alpha: R(G) \to KU^0(BG)$ and $R(G) \to \hat{R}(G)$.

If $Sp$ and $U$ are the “big” symplectic and unitary groups, the standard inclusion $l: Sp \to U$ induces a monomorphism $l^*: KSp^*(BSp(n)) \to KU^*(BSp(n))$ of abelian groups. An element of $KU^*(BSp(n))$ is called symplectic if it is in the image of $l^*$. 

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If $\theta: Sp(n) \to Sp(n)$ is the identity representation, then $\alpha(\theta - 2n) = y_1$. Thus $y_1$ is symplectic and hence so is $f^*y_1$.

**Lemma 2.** The image of $I^*: KSp^0(BSp(1)) \to KU^0(BSp(1))$ is the subgroup generated by $\{1, y, 2y^2, \ldots, y^{2i-1}, 2y^{2i}, \ldots\}$.

**Proof.** Since $y$ is symplectic, so is $y^{2i-1}$ and since $y^{2i}$ is self-conjugate, $2y^{2i}$ is symplectic. From [7] we deduce that $y^{2i}$ is not symplectic.

So, if $f^*y_1 = \Sigma y(r)y^r$, then $y(2r)$ is even and our 2-primary restrictions on $\{m(j)\}$ arise from this fact.

**Proposition 3.**

$$y(r) = \sum_{j=1}^{n} \frac{m(j)}{r} \left( \frac{m(j) + r - 1}{2r - 1} \right),$$

where $(\cdot)$ is the binomial coefficient.

**Corollary 4.** For each integer $r \geq 1$,

$$\sum_{j=1}^{n} \frac{m(j)}{2r} \left( \frac{m(j) + 2r - 1}{4r - 1} \right)$$

is even.

**Proof.** This is the condition that $y(2r)$ is even.

The proof of Proposition 3 needs

**Lemma 5.** Proposition 3 is true for $n = 1$, i.e.

$$y(r) = \frac{m}{r} \left( \frac{m + r - 1}{2r - 1} \right), \quad m(1) = m.$$

**Proof.** We have the Adams operation $\psi^2: KU^0(BSp(1)) \to KU^0(BSp(1))$ [1]. It is natural and $\psi^2z_i = z_i^2$, $\psi^2y = 4y + y^2$.

The naturality of $\psi^2$ gives

$$\sum y(r)(4y + y^2)^r = 4f^1y + (f^1y)^2.$$

From Proposition 1 and [3] we know a priori that $f^1 = \psi^m$. Thus computing $f^1$ amounts to writing $z^m + z^{-m} - 2$ as a polynomial in $(z + z^{-1} - 2)$. This can be done with the help of $(\ast)$.

**Proof of Proposition 3.** This follows from Lemma 5 and the equation $\text{Ch} f^1y_1 = f^*\text{Ch} y_1$, where Ch is the Chern character [4].

**Lemma 6.** For any integers $m$ and $m'$ let $m = \Sigma \beta_i 2^i$ and $m' = \Sigma \beta_i' 2^i$ be their $2$-adic expansions with $\beta_i, \beta_i' = 0$ or 1. Then

$$\left( \begin{array}{c}
m \\
m'
\end{array} \right) \equiv \prod_i \left( \begin{array}{c}
\beta_i \\
\beta_i'
\end{array} \right) \mod 2.$$
Proof. Well known.

To get our information on \( \{m(i)\} \), we need more notation.

**Definition.** (i) For any integer \( m \neq 0 \), write \( m = 2^s m' \), where \( m' \) is odd and put \( \beta(m) = s \).

(ii) Divide \( \{m(i)\} \) into disjoint subsets \( I_0, I_1, \ldots \), such that \( a \in I_\beta \Rightarrow \beta(a) = s \).

(iii) If in \( \{m(j)\} \), \( m(i) \) is repeated \( d(i) \) times and \( I_s \) contains the distinct elements \( m(j_1), m(j_2), \ldots \), define \( \text{Card} I_s \) to be \( d(j_1) + d(j_2) + \ldots \).

(iv) Put

\[
C_i(r) = \binom{m(i) + r - 1}{2r - 1}.
\]

**Lemma 7.**

(i) \[
C_i(r) = \binom{2m(i)}{m(i) + r} \binom{m(i) + r}{2r}.
\]

(ii) If \( \beta(r) = s \) and \( m(i) \notin I_{s+1} \), then \( C_i(2r) \) is even.

**Proof.** (i) Easy from the definition of the binomial coefficient.

(ii)

\[
\beta(m(i)/(m(i) + 2r)) = \beta(m(i)) - \beta(m(i) + 2r) = \beta(m(i)) - (s + 1) \geq 0 \text{ if } \beta(m(i)) > s + 1,
\]

\[
= 0 \text{ if } \beta(m(i)) < s + 1.
\]

In either case, \( \beta(2m(i)/(m(i) + 2r)) > 0 \), and hence \( C_i(2r) \) is even.

**Proposition 8.** (i) If \( I_s \) is not empty then \( s > 0 \Rightarrow \text{Card} I_s \) is even.

(ii) If \( I_s \) is not empty and \( s > 0 \), let the distinct elements of \( I_s \) for which \( \beta(\cdot) \) is odd be \( m(1), \ldots, m(e^* \cdot) \). Then (by part (i)) \( e^* = 2e \) and there are integers \( w_i \) and \( b_i \) with \( b_i = 0 \) or \( 1 \) such that

\[
m(2i - 1) = 2^s(1 + 4w_i + 2b_{2i-1}),
\]

\[
m(2i) = 2^s(1 + 4w_i + 2b_{2i}), \quad i = 1, \ldots, e.
\]

**Proof.** (i) By renumbering if necessary, we can assume that the distinct integers in \( I_s \) are the first \( e' \) from \( \{m(i)\} \). Write \( m(i) \) as \( m(i) = \sum_{u \geq 0} a(iu) 2^{u+s} \), \( a(i0) = 1 \), \( a(iu) = 0 \) or \( 1 \), and \( 1 \leq i \leq e' \).

Let \( r = 2^s-1 + b_1 2^s + \ldots \). Then Lemma 7 implies that \( C_i(2r) \) is even if \( m(i) \notin I_s \), hence Corollary 4 gives

\[
\sum_{i \in e'} d(i)C_i(2r) = 0 \mod 2.
\]

Since \( \beta(m(i)/2r) = 0 \), this gives
\[ \sum d(i) \left( \frac{m(i) + 2r - 1}{4r - 1} \right) = 0 \mod 2. \]

From Lemma 6 we have
\[ \left( \frac{m(i) + 2r - 1}{4r - 1} \right) = \left( \frac{b_2 + a(i2)}{b_1} \right) \left( \frac{b_3 + a(i3)}{b_2} \right) \cdots . \]

Choose \( b_i = 0 \) for each \( i \). Then all binomial coefficients in the above line become 1. Hence
\[ \sum_{1 \leq i \leq e} d(i) = 0 \mod 2. \]

This proves (i) since the left-hand side is \( \text{Card} I_s \).

(ii) We can assume that the distinct \( m(i) \) in \( I_s \) with odd \( d(\cdot) \) are the first \( 2e \) from \( \{m(j)\} \). From the proof of (i), it is clear that since we are assuming the \( d(i) \) to be odd, the information we have is
\[ (**) \quad \sum_{i=1}^{2e} a(ik_1) \cdots a(ik_v) = 0 \mod 2, \quad v \geq 1, \quad 2 \leq k_1 < \cdots < k_v. \]

When \( e = 1 \), take \( v = 1 \) in (**) to get \( a(1u) = a(2u), u > 1 \). To "solve" (**) in general we need

Lemma 9. Consider the following system over \( \mathbb{Z}_2 \):
\[ (**) \quad \sum_{i=1}^{2e} a(i, k_1) \cdots a(i, k_v) = 0, \quad 2 \leq k_1 < \cdots < k_v. \]

This system is satisfied \( \iff \) the \( a(\cdot, k) \) are equal in pairs i.e. for each \( i \), there is an \( i', i \neq i' \), such that \( a(i, k) = a(i', k) \) for all \( k \geq 2 \).

Proof. \( \Leftarrow \) Obviously the system is satisfied if \( a(i, k) = a(i', k) \).

\( \Rightarrow \) Conversely, we solve (**) by induction on \( e \). The conclusion of the lemma is true for \( e = 1 \). Let the conclusion be true for systems
\[ (**) \quad \sum_{i=1}^{2e'} a'(i, k_1) \cdots a'(i, k_v) = 0, \quad e > e', \quad 2 \leq k_1 < \cdots < k_v. \]

If in (**) the \( a's \) are all 0 or all 1, we are finished. Assume therefore that the \( a(i, 2) \) are not all equal. Clearly we can assume without loss of generality that
\[ a(1, 2) = \cdots = a(2q, 2) = 1, \quad a(2q + 1, 2) = \cdots = a(2e, 2) = 0 \]
for some \( q < e \).

In (**) take \( k_1 = 2 \). We get
\[ \sum_{1 \leq i \leq 2q} a(i, k_1) \cdots a(i, k_v) = 0, \quad 3 \leq k_2 < \cdots < k_v. \]
By the induction hypothesis, for each $i$, there is an $i'$ ($1 \leq i, i' \leq 2q$) such that $a(i, k) = a(i', k), k \geq 3$. Hence from (**) we get
\[ \sum_{2q+1 \leq i \leq 2e} a(i, k_2) \cdots a(i, k_v) = 0, \]
and by the induction hypothesis, for each $i$, there is an $i'$ ($2q + 1 \leq i, i' \leq 2e$) such that $a(i, k) = a(i', k), k \geq 3$. This completes the proof of Lemma 9.

To complete the proof of Proposition 8(ii), take
\[ b_i = a(i1), \quad 1 \leq i \leq 2e \quad \text{and} \quad w_j = \sum_{u \geq 2} a(2j - 1, u)2u + s, \quad 1 \leq j \leq e. \]

Finally, we have our 2-primary result on $\{m(j)\}$.

**Theorem.** With the notation of Proposition 8, each element of $I_s, s > 0$, has an even $d(\ )$, i.e. each element occurs an even number of times.

**Proof.** This is an easy corollary of Proposition 8, for $f^1 \psi^3 y_1$ is symplectic [2, p. 71]. Thus the argument of Proposition 8 gives
\[ 3m(2i - 1) = 2^s(1 + 4w'_i + 2b'_{2i - 1}), \]
\[ 3m(2i) = 2^s(1 + 4w'_i + 2b'_{2i}) \quad \text{for some} \ w'_i \text{ and } b'_{2i}. \]

Thus $m(2i) = m(2i - 1)$. Hence $e^* = 0$.

In summary, our 2-primary restriction is that in $\{m(j)\}$, each even $m(j)$ occurs an even number of times.

**Corollary.** If all the $m(i)^2$ are equal, to $m^2$ say, then $n$ odd $\Rightarrow m$ odd or zero.

**Proof.** Let $m \in I_s$. If $s > 0$, then $\text{Card } I_s = n$ is even by the Theorem.

Notes. (1) The case $n = 1$ is given in [6].

(2) It is clear from the Theorem that using our method, $KSp$ will not give further information on $\{m(j)\}$.

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