CLASSIFICATION OF HOMOTOPY TORUS KNOT SPACES

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ABSTRACT. The existence of nontrivial homotopy torus knot spaces is established as a corollary to the

Theorem. Let $p$ and $q$ be two integers with $p > 1$, $q > 1$, and $(p, q) = 1$. Let $\mathcal{M}$ be a maximal set of topologically distinct compact orientable irreducible 3-manifolds with fundamental group presented by $(a, b|a^pb^q)$. Then $\text{card}(\mathcal{M}) = \frac{1}{2}\phi(pq)$, where $\phi$ denotes Euler's function.

All spaces are piecewise linear. The symbols $I$, $D$, and $B$ denote the closed unit interval, disc, and ball, respectively. $S^i$ denotes the $i$-dimensional sphere; $i = 1, 2, 3$. The closure and boundary of a space $X$ are denoted respectively by $\text{cl}(X)$ and $\partial X$. The term knot space refers to the closure of the complement in $S^3$ of a regular neighborhood of a knot. If $m$ and $n$ are positive integers and $(m, n) = 1$, then the torus knot space corresponding to the pair $(m, n)$ is denoted by $K_{m, n}$.

A 3-manifold is irreducible if every embedded 2-sphere bounds a 3-cell. Let $M$ be a compact manifold with boundary and $K$ be a 2-complex in $M$. If $M - K$ is homeomorphic to $\partial M \times (0, 1]$, then $K$ is called a spine of $M$. If $M$ has empty boundary and $K$ is a spine of $\text{cl}(M - B)$, then we will say that $K$ is a spine of $M$.

If $M_1$ and $M_2$ are compact 3-manifolds with boundary, $M_1 \subset M_2$, $\text{cl}(M_2 - M_1) = U = D \times I$, and $M_1 \cap U = \partial D \times I$, then we will say that $M_2$ is obtained from $M_1$ by attaching the 2-handle $U$ to $M_1$. If $\gamma$ is a simple closed curve in $\partial M_1$ having $\partial D \times I$ as a regular neighborhood in $\partial M_1$, then we will say that $U$ was attached to $M_1$ along $\gamma$.

We distinguish between the terms group and group presentation. If $\psi = (x_1, \ldots, x_m|R_1, \ldots, R_n)$ is a group presentation, then $G_{\psi}$ denotes the group presented by $\psi$, $K_{\psi}$ the 2-complex corresponding to $\psi$, and $P_{\psi}$ the corresponding $P$-graph (see [4]). Note that if $K_{\psi}$ is a spine of the compact 3-manifold $M$, then $\pi_1(M) = G_{\psi}$.

Henceforth $\psi$ denotes the presentation $(a, b|a^pb^q)$.

Lemma. Under the conditions of the Theorem, if $M \in \mathcal{M}$ then $K_{\psi}$ is a spine of $M$.

Proof. By results of Waldhausen [5], $M$ is a Seifert fiber space with

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orientable quotient surface $E$, and $\pi_1(M)$ is presented by

$$\psi^* = (x_1, \ldots, x_r, y_1, \ldots, y_s, z \mid [x_i, z] = 1, y_i^{\mu_i} = z),$$

where the genus of $E$ is $g \geq 0$, where $\partial M$ has $r + 1 > 0$ components, each component a torus $S^1 \times S^1$, and where $M$ has $s > 0$ exceptional fibers with respective orders $\mu_1, \ldots, \mu_s$. Since $G_\psi$ is isomorphic to $G_{\psi^*}$, we infer that $r = g = 0, s = 2$, and that $\mu_1$ and $\mu_2$ are $p$ and $q$ in some order. This is argued by considering $\pi_1(M)$ modulo its center and applying results of [2, §4.1].

It now follows that $E$ is a disc with two exceptional points $x_1$ and $x_2$ corresponding to the two exceptional fibers. Let $\gamma$ be an arc in $E$ with $\partial \gamma = \gamma \cap \partial E$ and $\gamma$ separating $x_1$ from $x_2$. Let $F$ be the union of all fibers projecting onto $\gamma$. Then $F$ is an annulus that separates $M$ into two solid tori $T_1$ and $T_2$. Write $F = S^1 \times I$ and consider the arc $\delta = 1 \mid t \times I$ in $M$, where $t \in S^1$. Note that $\partial \delta = \delta \cap \partial M$. Let $U$ be a regular neighborhood of $\delta$. Then $\text{cl}(M - U)$ is a genus 2 handlebody to which the 2-handle $U$ is attached. Clearly $U \cap T_1$ and $U \cap T_2$ are each connected. Thus $U$ is attached to $T$ according to the word $a^pb^q$, and the Lemma follows.

**Proof of the Theorem.** Choose $r$ and $s$ so that $(r, p) = 1 \leq r < p$ and $(s, q) = 1 \leq s < q$. By the results of [3], we know that a compact orientable 3-manifold $M_{p, r, q, s}$ with boundary and with spine $K_\psi$ is uniquely determined by the faithful embedding of $P_\psi$ in $S^2$ with gap $r$ on the $a$-syllable graph and gap $s$ on the $b$-syllable graph. Any such manifold, having $K_\psi$ as a spine, is thus of the same homotopy type as $K(p, q)$. It will suffice to show that $M_{p, r, q, s}$ and $M_{p', r', q', s'}$ are homeomorphic if and only if $r \equiv \epsilon r' \pmod{p}$ and $s \equiv \epsilon s' \pmod{q}$ where $\epsilon = \pm 1$.

The if part is clear; if $\epsilon = -1$, then we merely reverse the orientation.

Let $T$ denote a genus 2 handlebody with inner meridian discs corresponding to the generators $a$ and $b$. Following the construction of $M = M_{p, r, q, s}$, the above-mentioned faithful embedding of $P_\psi$ determines a simple closed curve $\gamma_0$ (corresponding to the word $a^pb^q$) in $\partial T$. Then $M$ is obtained from $T$ by attaching a 2-handle $U$ along $\gamma_0$, and $\partial M$ is homeomorphic to the torus $S^1 \times S^1$.

One can construct simple closed curves $\gamma_1$ and $\gamma_2$ in $\partial T - U$ that correspond respectively to the words $a^p$ and $b^s$. Moreover this can be done so that $\partial M - (\gamma_1 \cup \gamma_2)$ is connected, $\gamma_1$ and $\gamma_2$ intersecting in a single crossing point.

To see this, we use the techniques of [3]. Construct in $S^2$ a faithfully embedded $a$-syllable graph with three syllables whose exponents are $p, p$, and $r$, respectively, and a faithfully embedded $b$-syllable graph with two syllables whose exponents are $q$ and $s$. Note that this is possible since the gaps on
the syllables \( a^p \) and \( b^q \) are \( r \) and \( s \) respectively. See Figure 1 for the construction. The ends of the \( a \)-syllables are indicated by \( 0, 1, \) and \( 2 \), and those of the \( b \)-syllables by \( 0 \) and \( 2 \). Connect the syllable ends with arcs as shown. In constructing \( T \), we obtain simple closed curves \( \gamma_0, \gamma_1, \) and \( \gamma_2 \) in \( \partial T \) corresponding, respectively, to the words \( a^p b^q, a^p, \) and \( a^r b^s \). Moreover \( \gamma_1 \) and \( \gamma_2 \) intersect in the point \( Q \), which is clearly a crossing point. Attaching the 2-handle \( U \) to \( T \) along \( \gamma_0 \) gives us the manifold \( M \) with the curves \( \gamma_1 \) and \( \gamma_2 \) in \( \partial M - U \) and intersecting at \( Q \).

![Figure 1](image-url)

We show that \( \gamma_1 \cup \gamma_2 \) does not separate \( \partial M \). Let

\[
\psi_1 = \langle a, b \mid a^p b^q, a^p \rangle \quad \text{and} \quad \psi_2 = \langle a, b \mid a^p b^q, a^r b^s \rangle.
\]

Then \( K_{\psi_1} \) and \( K_{\psi_2} \) are spines of closed manifolds—the former of a connected sum of two lens spaces (by the multiplication theorem \([4]\)) and the latter of a lens space (see \([4]\)). This means that each \( M_i \) has a 2-sphere boundary; hence, each curve \( \gamma_i \) does not separate \( \partial M \) \((i = 1, 2)\). It follows that \( \gamma_1 \cup \gamma_2 \) does not separate \( \partial M \).

Let \( y \) be any nonseparating simple closed curve in \( \partial M \). Since \( \pi_1(\partial M) \) is abelian and is generated by \( a^p \) and \( a^r b^s \), we observe that \( y \) corresponds to the word \( W = (a^p)^m(a^r b^s)^n \) in \( \pi_1(M) \) for an appropriate choice of \( m \) and \( n \) \((m, n) = 1\). Let \( \widehat{M} \) be a closed manifold obtained from \( M \) by attaching a 2-handle \( U \) to \( M \) along \( y \) and then attaching a 3-cell to the 2-sphere boundary of the resulting manifold. Then \( \pi_1(\widehat{M}) \) is presented by \( \widehat{\psi} = \langle a, b \mid a^p b^q, W \rangle \).

We show that \( \widehat{M} \) is a lens space if and only if \( |n| = 1 \). If \( n = 0 \), then \((m, n) = 1 \) forces \( m = 1 \) and \( \widehat{M} \) is a connected sum of two nontrivial lens spaces. If \( |n| > 1 \), then \( \pi_1(\widehat{M}) \) has a homomorphism onto the group present-
ed by \( (a, b | a^p = b^q = (a^r b^s)^n = 1) \). This group can be shown not to be cyclic (see [1, p. 71]).

If \(|n| = 1\), we assume \( n = 1 \) and obtain \( \hat{\psi} = \langle a, b | a^p b^q, a^{mp + r} b^s \rangle \) with \( K_{\hat{\psi}} \) a spine of \( \hat{M} \). Thus, \( \hat{M} \) is a lens space. Let \( \lambda \) be the order of \( \pi_1(\hat{M}) \). Then \( \lambda = |ps - q(mp + r)| \). Thus, assuming that \( M_{p, r, q, s} \) and \( M_{p', r', q, s'} \) are homeomorphic, we conclude that

\[
ps - qr - mpq = \epsilon(ps - qr - m'pq)
\]

for \( \epsilon = \pm 1 \) and appropriate choices of \( m \) and \( m' \). Hence,

\[
p(s - cs') - q(r - cr') = pq(m - cm'),
\]

and the theorem follows.

**Corollary.** There exists a compact orientable irreducible 3-manifold which is not embeddable in \( S^3 \) but which is of the same homotopy type as a torus knot space.

**Proof.** Suppose that \( M = M_{p, r, q, s} \) is embeddable in \( S^3 \). Then \( \partial M \) is a torus \( S^1 \times S^1 \) in \( S^3 \). Since \( M \) is not a solid torus, it follows that \( \text{cl}(S^3 - M) \) is a solid torus. Hence \( S^3 \) is obtainable from \( M \) by attaching a 2-handle along some nonseparating simple closed curve in \( \partial M \) and then attaching a 3-cell to the resulting 2-sphere boundary. Thus \( ps - qr - mpq = \pm 1 \) for some \( m \), a condition that is violated for \( p = 5, q = 3, r = 1, \) and \( s = 1 \).

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**REFERENCES**


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