PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, October 1975

SIMPLE KNOTS WHICH ARE DOUBLY-NULL-COBORDANT

C. KEARTON

ABSTRACT. Using the Seifert matrix, a necessary and sufficient condition is given for a simple $(2q-1)$-knot, $q>1$, to be doubly-null-cobordant.

An *n-knot* is a smooth oriented submanifold K of the $(n+2)$-sphere S^{n+2}, where K is homeomorphic to S^n. An n-knot is *doubly-null-cobordant* if it is a cross-section of an unknotted pair (S^{n+3}, S^{n+1}). D. W. Sumners [3] has studied the case $n=2q-1$, and has given a necessary condition for K to be doubly-null-cobordant. He has also given a partial converse, which it is the purpose of this paper to strengthen.

Let K be a $(2q-1)$-knot; K bounds a smooth oriented submanifold U of S^{2q+1}, and choosing a basis for the torsion-free part of $H_q(U)$, we obtain a matrix A of linking numbers called a Seifert matrix of K. The submanifold U is not unique, but any two such submanifolds are cobordant, and this induces an equivalence relation on the Seifert matrices of K which may be described as follows. An *elementary enlargement* of A is an integer matrix of the form

$$\left(\begin{array}{c|cc} A & 0 & 0 \\ \hline \alpha & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cc} A & \beta & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

where α is a row vector, β a column vector. We say $A \sim B$ if A is an elementary enlargement of B or vice versa, or if there is a unimodular congruence between A and B. Two Seifert matrices A, B are *equivalent* if they are connected by a sequence $A = A_1, \ldots, A_k = B$ such that $A_i \sim A_{i+1}$ for $1 \leq i < k$.

An elementary enlargement is *trivial* if $\alpha = 0$ or $\beta = 0$, and a succession of trivial elementary enlargements is a *trivial enlargement.*

The Seifert matrix A is *doubly-null-cobordant* if it is congruent to one of the form $\left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$, all the blocks being square.

Levine [2] defines a *simple* $(2q-1)$-knot as one whose complement has the homotopy $(q-1)$-type of a circle. If K is a $(2q-1)$-knot which bounds a $(q-1)$-connected submanifold U, then the associated matrix A is said to be *special*; we remark that this can occur precisely when K is simple.

Received by the editors July 12, 1974.

AMS (MOS) subject classifications (1970). Primary 57C45, 55A25; Secondary 57D40.

Key words and phrases. Simple knot, doubly-null-cobordant, Seifert matrix.

Lemma. *Let A be a Seifert matrix of a simple $(2q-1)$-knot K. If $q > 2$, then A is special; if $q = 2$, then some trivial enlargement of A is special.*

Proof. Let U be a submanifold with which A is associated. By results of [1], U is cobordant to a $(q-1)$-connected submanifold V, and so there exists a sequence $A = A_1, \ldots, A_k = B$ where $A_i \sim A_{i+1}$ for $1 \leq i < k$, and B is a Seifert matrix associated with V. If $q > 2$, it is shown in [2, §§11–13] that if $C \sim D$ and D is a special Seifert matrix, then so is C. But B is special, hence so is A. If $q = 2$, the arguments in [2, §§10–13] apply provided we replace the sequence $A_1, \ldots, A_k$ by $A_1', \ldots, A_k'$, each A_i' being a trivial enlargement of A_i with order A_i' – order A_i independent of i. □

Theorem. *If K is a simple $(2q-1)$-knot, $q \geq 2$, then K is doubly-null-cobordant if and only if it possesses a doubly-null-cobordant Seifert matrix.*

Proof. If K is doubly-null-cobordant, then Sumners proves in [3] that it has a doubly-null-cobordant matrix. Assume conversely that A is such a matrix; then any trivial enlargement of A is also doubly-null-cobordant, and so by the Lemma we may take A to be special. Another result of Sumners [3, Theorem 3.1] shows that K is doubly-null-cobordant. □

REFERENCES

1. Jerome Levine, *Unknotting spheres in codimension two*, Topology 4 (1965), 9–16. MR 31 #4045.

2. ———, *An algebraic classification of some knots of codimension two*, Comment. Math. Helv. 45 (1970), 185–198. MR 42 #1133.

3. D. W. Sumners, *Invertible knot cobordisms*, Comment. Math. Helv. 46 (1971), 240–256. MR 44 #7535.

CORPUS CHRISTI COLLEGE, CAMBRIDGE UNIVERSITY, CAMBRIDGE, GREAT BRITAIN