SIMPLE KNOTS WHICH ARE DOUBLY-NULL-COBORDANT

C. KEARTON

ABSTRACT. Using the Seifert matrix, a necessary and sufficient condition is given for a simple $(2q - 1)$-knot, $q > 1$, to be doubly-null-cobordant.

An $n$-knot is a smooth oriented submanifold $K$ of the $(n + 2)$-sphere $S^{n+2}$, where $K$ is homeomorphic to $S^n$. An $n$-knot is doubly-null-cobordant if it is a cross-section of an unknotted pair $(S^{n+3}, S^{n+1})$. D. W. Sumners [3] has studied the case $n = 2q - 1$, and has given a necessary condition for $K$ to be doubly-null-cobordant. He has also given a partial converse, which is the purpose of this paper to strengthen.

Let $K$ be a $(2q - 1)$-knot; $K$ bounds a smooth oriented submanifold $U$ of $S^{2q+1}$, and choosing a basis for the torsion-free part of $H_q(U)$, we obtain a matrix $A$ of linking numbers called a Seifert matrix of $K$. The submanifold $U$ is not unique, but any two such submanifolds are cobordant, and this induces an equivalence relation on the Seifert matrices of $K$ which may be described as follows. An elementary enlargement of $A$ is an integer matrix of the form

\[
\begin{pmatrix}
A & 0 & 0 \\
\alpha & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
A & \beta & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

where $\alpha$ is a row vector, $\beta$ a column vector. We say $A \sim B$ if $A$ is an elementary enlargement of $B$ or vice versa, or if there is a unimodular congruence between $A$ and $B$. Two Seifert matrices $A, B$ are equivalent if they are connected by a sequence $A = A_1, \ldots, A_k = B$ such that $A_i \sim A_{i+1}$ for $1 \leq i < k$.

An elementary enlargement is trivial if $\alpha = 0$ or $\beta = 0$, and a succession of trivial elementary enlargements is a trivial enlargement.

The Seifert matrix $A$ is doubly-null-cobordant if it is congruent to one of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, all the blocks being square.

Levine [2] defines a simple $(2q - 1)$-knot as one whose complement has the homotopy $(q - 1)$-type of a circle. If $K$ is a $(2q - 1)$-knot which bounds a $(q - 1)$-connected submanifold $U$, then the associated matrix $A$ is said to be special; we remark that this can occur precisely when $K$ is simple.

Received by the editors July 12, 1974.


Key words and phrases. Simple knot, doubly-null-cobordant, Seifert matrix.
Lemma. Let $A$ be a Seifert matrix of a simple $(2q - 1)$-knot $K$. If $q > 2$, then $A$ is special; if $q = 2$, then some trivial enlargement of $A$ is special.

Proof. Let $U$ be a submanifold with which $A$ is associated. By results of [1], $U$ is cobordant to a $(q - 1)$-connected submanifold $V$, and so there exists a sequence $A = A_1, \ldots, A_k = B$ where $A_i \sim A_{i+1}$ for $1 \leq i < k$, and $B$ is a Seifert matrix associated with $V$. If $q > 2$, it is shown in [2, §§11–13] that if $C \sim D$ and $D$ is a special Seifert matrix, then so is $C$. But $B$ is special, hence so is $A$. If $q = 2$, the arguments in [2, §§10–13] apply provided we replace the sequence $A_1, \ldots, A_k$ by $A'_1, \ldots, A'_k$, each $A'_i$ being a trivial enlargement of $A_i$ with order $A'_i - \text{order } A_i$ independent of $i$. □

Theorem. If $K$ is a simple $(2q - 1)$-knot, $q \geq 2$, then $K$ is doubly-null-cobordant if and only if it possesses a doubly-null-cobordant Seifert matrix.

Proof. If $K$ is doubly-null-cobordant, then Sumners proves in [3] that it has a doubly-null-cobordant matrix. Assume conversely that $A$ is such a matrix; then any trivial enlargement of $A$ is also doubly-null-cobordant, and so by the Lemma we may take $A$ to be special. Another result of Sumners [3, Theorem 3.1] shows that $K$ is doubly-null-cobordant. □

REFERENCES


CORPUS CHRISTI COLLEGE, CAMBRIDGE UNIVERSITY, CAMBRIDGE, GREAT BRITAIN