FINITE UNIONS OF IDEALS AND MODULES

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ABSTRACT. We say that a commutative ring \( R \) is a \( u \)-ring provided \( R \) has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a \( um \)-ring is a ring \( R \) with the property that an \( R \)-module which is equal to a finite union of submodules must be equal to one of them. The primary purpose of this paper is to characterize \( u \)-rings and \( um \)-rings. We show that \( R \) is a \( um \)-ring if and only if the residue field \( R/P \) is infinite for each maximal ideal \( P \) of \( R \); and \( R \) is a \( u \)-ring if and only if for each maximal ideal \( P \) of \( R \) either the residue field \( R/P \) is infinite or the quotient ring \( R_P \) is a Bézout ring.

Introduction. All rings considered are commutative with identity \( 1 \neq 0 \), and \( R \) denotes a ring with total quotient ring \( T(R) \). For a multiplicative system \( S \) of \( R \) and a unitary \( R \)-module \( M \), we use standard conventions concerning properties and notation for the quotient ring \( R_S \) and the quotient module \( M_S \) (e.g. see [G, p. 52], [N, p. 14], [K, p. 22], [ZS, p. 221]; in particular, \( AR_S \) will denote the ideal in \( R_S \) associated with the ideal \( A \) of \( R \) by the canonical map from \( R \) to \( R_S \)). The "running indices" will usually be dropped from finite intersections, unions, and sums; thus \( \bigcup A_i, \bigcap A_i, \) and \( \Sigma A_i \) will uniformly mean that \( i \) ranges from \( i = 1 \) to \( i = n \). An ideal \( A \) of \( R \) is called a \( u \)-ideal provided \( A \subset \bigcup A_i \) implies \( A \subset A_{i_0} \) for some \( i_0 \), where \( A_1, \ldots, A_n \) are ideals of \( R \).

It is well known that an ideal of \( R \) contained in a finite union of prime ideals of \( R \) must be contained in one of those primes [K, p. 55], [M], but it does not seem to have been generally observed that this property holds in certain rings (e.g. Dedekind domains) even if none of the ideals involved are prime. In [M], McCoy shows that \( A \subset \bigcup A_i \) implies \( A^k \subset \bigcap A_i \) for some positive integer \( k \) in a general commutative ring, provided \( A \) is not contained in the union of any \( n-1 \) of the \( A_i \); in addition he proves an analogous result for subgroups of a group \( G \).

1. Preliminary results. In this section we show that in considering \( u \)-rings we can replace \( A \subset \bigcup A_i \) by \( A = \bigcup A_i \), thus motivating the definition of \( um \)-rings; moreover, only finitely generated \( A \) need be considered. In addition, we show that every invertible ideal is a \( u \)-ideal.

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The proofs of the following two propositions are easy and we omit them.

**Proposition 1.1.** The following conditions are equivalent, for $A, A_1, \ldots, A_n$ ideals of $R$.

1. Each ideal of $R$ is a u-ideal.
2. Each finitely generated ideal of $R$ is a u-ideal.
3. $A = \bigcup A_i$ finitely generated $\Rightarrow A = A_i$ for some $i$.
4. $A = \bigcup A_i \Rightarrow A = A_i$ for some $i$.

**Corollary 1.2.** Every Bezout ring (i.e., a ring in which every finitely generated ideal is principal) is a u-ring.

**Proposition 1.3.** If $R$ is a u-ring (um-ring), then every homomorphic image and every quotient ring $R_\mathfrak{p}$ of $R$ is a u-ring (um-ring).

It is clear that a um-ring is a u-ring, but the converse is false (e.g., the ring of integers). It is easy to check that if $M = \bigcup M_i$ implies $M = M_i$ for some $i$, where $M$ is a finitely generated $R$-module and $M_1, \ldots, M_n$ are submodules of $M$, then $R$ is a u-ring.

**Proposition 1.4.** If $(0)$ is the annihilator of the finitely generated ideal $A$ of $R$ and $B_i \neq R$ is an ideal of $R$ for $i = 1, \ldots, n$, then $A \notin \bigcup A_i$.

**Proof.** We use induction on $n$. Consider $n = 1$ and suppose $A = AB_1$. There exists $b \in B_1$ such that $a = ab$ for all $a \in A$ ([ZS, p. 215]), implying $B_1 = R$, a contradiction. Suppose the proposition holds for $n - 1 > 1$ and consider two cases.

Case 1. Among $B_1, \ldots, B_n$ there are two ideals, say $B_1$ and $B_2$, such that $B_1 + B_2 \neq R$. Since $A \notin A(B_1 + B_2) \cup AB_3 \cup \ldots \cup AB_n$, it follows that $A \notin \bigcup AB_i$, completing Case 1. In Case 2, suppose $B_i + B_j = R$ for all $i \neq j$. Let $C_i = B_i \cap B_j$ (if $i \neq j$) for $i = 1, \ldots, n$. Then $B_i + C_i = R$ ([ZS, p. 177]) and $AB_i + AC_i = A$ for $i = 1, \ldots, n$. If $AC_i \subset AB_i$ for some $i$, then $A = AB_i + AC_i = AB_i$, which is impossible by the case $n = 1$. Hence there exists $a_i \in AC_i \setminus AB_i$ for $i = 1, \ldots, n$. Set $a = \sum a_i$ and note that $a \in A \setminus AB_i$ for $i = 1, \ldots, n$, completing the proof.

By a fractional ideal $F$ of $R$ we understand an $R$-module $F \subset T(R)$ for which there exists a regular element (i.e., not a zero divisor) $r$ of $R$ such that $rF \subset R$.

**Theorem 1.5.** Every invertible ideal of $R$ is a u-ideal.

**Proof.** If $A$ is an invertible ideal and $A_1, \ldots, A_n$ are ideals of $R$ such that $A \subset \bigcup A_i$, then $A = \bigcup C_i$ where $C_i = A \cap A_i$. Suppose $A \neq C_i$ for $i = 1, \ldots, n$. Then $C_i = AB_i$ for each $i$, where $R \neq B_i = C_i A^{-1}$ is an ideal of $R$. Since $A$ is invertible, the annihilator of $A$ is $(0)$ and $A$ is finitely generated ([ZS, p. 272]). Proposition 1.4 implies $A \notin \bigcup AB_i = \bigcup C_i$, a contradiction.
Corollary 1.6. Every Prüfer domain is a u-domain.

Griffin [MG] defines a Prüfer ring as a ring in which every finitely generated regular ideal is invertible; this type of Prüfer ring need not be a u-ring as can be seen by taking $R = \mathbb{Z}[x]/(2, x)^2$, where $\mathbb{Z}[x]$ is the polynomial ring over the integers. Butts and Smith [BS] called $R$ a Prüfer ring provided the ideals of $R_p$ are linearly ordered by inclusion for every proper prime ideal $P$ of $R$; it follows from the characterization given in §2 that Prüfer rings of this type are u-rings.

Proposition 1.7. If $R$ contains an infinite set $S$ such that $x - y$ is a unit of $R$ for all $x \neq y$ in $S$, then $R$ is a um-ring.

Proof. Suppose $M_1, \ldots, M_n$ are submodules of an $R$-module $M$ such that $M = \bigcup M_i$ and $M \neq M_i$ for each $i$. It is no restriction to assume for each $i$ that $M_i \neq \bigcup M_j (j \neq i)$, so let $m_i \in M_i$ and $m_i \notin \bigcup M_j (j \neq i)$ for $i = 1, 2$ and consider the set $E = \{m_1 + x m_2 | x \in S\}$. There exist $x$ and $y$ in $S$ such that $x \neq y$ and both $m_1 + x m_2$ and $m_1 + y m_2$ belong to the same $M_i$ for some $i \neq 2$, implying $m_2 \in M_i$, a contradiction.

Theorem 1.8. If there exists a maximal ideal $P$ of $R$ such that $F = R/P$ is finite with $n - 1$ elements, then for an $R$-module $M$ (ideals of $R$) such that the vector space $V = M/PM$ is not 1-dimensional over $F$ there exist $n$ submodules (ideals of $R$) $M_i \subset M$ such that $M \neq M_i$ for $i = 1, \ldots, n$ and $M = \bigcup M_i$.

Proof. Let $B$ be a basis for $V$ over $F$ and let $b_1 \neq b_2$ be two elements of $B$. Consider the following subsets of $V$: $E_1 = B \setminus \{b_1\}$, $E_2 = B \setminus \{b_2\}$, and $E_{2+i} = (B \setminus \{b_1, b_2\}) \cup \{b_1 + x_i b_2\}$ where $x_i (i = 1, \ldots, q - 1)$ ranges over the nonzero elements of $F$. Let $S_j$ be the subspace of $V$ generated by $E_j$ for $j = 1, \ldots, n$. A routine argument shows that $V = \bigcup S_i$ and $S_i \neq V$ for $i = 1, \ldots, n$. Hence, if $f : M \to V$ is the canonical homomorphism and $M_i = f^{-1}(S_i)$ for $i = 1, \ldots, n$, $M = \bigcup M_i$ and $M \neq M_i$ for each $i$.

Corollary 1.9. Let $R$ be a quasi-local ring with maximal ideal $P$.

(i) If $R$ is a um-ring, then $R/P$ is infinite.

(ii) If $R$ is a u-ring, then either $R/P$ is infinite or $R$ is a Bézout ring.

Proof. If $M$ is finitely generated and not principal, then so is $M/PM$ as a vector space over $R/P$ [N, p. 13]; moreover, $M = R \oplus R$ is such a module.

2. Characterization of u-rings and um-rings. We first reduce the general case to the case in which $R$ has only finitely many maximal ideals, and then deal with that situation.

Proposition 2.1. A ring $R$ is a um-ring (u-ring) if and only if $R_S$ is a um-ring (u-ring) for each multiplicative system $S$ of $R$ which is the complement of a finite union of maximal ideals of $R$. 
Proof. If $R$ is a um-ring (u-ring), then so is $R_S$ by Theorem 1.3. Now, consider the converse and suppose $M = \bigcup M_i$, where $M_i$ is a proper submodule of the $R$-module $M$ (ideal $M$ of $R$) for $i = 1, \ldots, n$. Let $m_i \in M \setminus M_i$ and let $P_i$ be a maximal ideal of $R$ containing the ideal $A_i = [M_i : m_i]_R$ for $i = 1, \ldots, n$. If $S$ denotes the complement of $\bigcup P_i$ in $R$, then $M_S = \bigcup (M_i)_S$ is clear and $M_S \neq (M_i)_S$ since $m_i \notin (M_i)_S$.

**Theorem 2.2.** If $R$ has only finitely many maximal ideals, say $M_1, \ldots, M_n$, then the following statements are equivalent.

(a) $R$ is a um-ring.

(b) $R_P$ is a um-ring for every maximal ideal $P$ of $R$.

(c) $R/P$ is infinite for every maximal ideal $P$ of $R$.

(d) There exists an infinite set $S$ in $R$ such that $x - y$ is a unit in $R$ for all $x \neq y$ in $S$.

Proof. (a) $\Rightarrow$ (b) by Proposition 1.3, (b) $\Rightarrow$ (c) by Corollary 1.9, and (d) $\Rightarrow$ (a) by Proposition 1.7. Consider (c) $\Rightarrow$ (d). Let $S_i$ be a complete set of representatives for the nonzero elements of $R/M_i$, and let $\{s_{ij} : j = 1, 2, \ldots, \infty\}$ be a denumerable subset of $S_i$ for $i = 1, \ldots, n$. For each $i$, there exists $x_j$ in $R$ such that $x_j \equiv s_{ij} \pmod{M_i}$ for $i = 1, \ldots, n$ [ZS, p. 177]. Denote by $S$ the set of all $x_j$ thus obtained and observe that for $x_r \neq x_s$ in $S$ we have $x_r - x_s \notin \bigcup M_i$ and $x_r - x_s$ is a unit of $R$.

**Theorem 2.3.** The following statements are equivalent.

(a) $R$ is a um-ring.

(b) $R_P$ is a um-ring for every maximal ideal $P$ of $R$.

(c) $R/P$ is infinite for every maximal ideal $P$ of $R$.

Proof. This follows directly from Propositions 1.3, 2.1, and Theorem 2.2.

**Lemma 2.4.** If $R$ has only finitely many maximal ideals and $R_M$ is a Bézout ring for each maximal ideal $M$ of $R$, then $R$ is a Bézout ring.

Proof. Let $M_1, \ldots, M_n$ be the maximal ideals of $R$ and for each $i = 1, \ldots, n$, choose $m_i \in M_j$ for $j \neq i$ and $m_i \notin M_i$. For a finitely generated ideal $A$ of $R$, there exists $a_i \in A$ such that $AR_M = a_i R_M$ for $i = 1, \ldots, n$. If $a = \sum a_i m_i$, then $AR_M = a R_M$ for $i = 1, \ldots, n$ since $a_i R_M \subseteq AR_M = a_i R_M$ implies "$a_i$ divides $a_j$" in $R_M$ and $a R_M = a_i u_i R_M$ where $u_i$ is a unit in $R_M$. Consequently $A = a R$.

**Theorem 2.5.** If $R$ has only finitely many maximal ideals, say $M_1, \ldots, M_r$, and $R_{M_i}$ is a Bézout ring for $i = 1, \ldots, n$ ($1 < n \leq r$) while $R/M_i$ is infinite for $i > n$, then $R$ is a u-ring.

Proof. We can assume $r = n$ since $r = n$ implies $R$ is a Bézout ring by Lemma 2.4 and therefore a u-ring. Suppose $A = (a_1, \ldots, a_m)$ and $A_1, \ldots, A_s$...
are ideals in $R$ with $A = A_1 \cup \ldots \cup A_s$. For $B = R \setminus \bigcup_i M_i$ and $E = R \setminus \bigcup_i M_i \ (i > n)$, $R_B$ is a Bezout ring by Lemma 2.4 and $R_E$ has the property that each maximal ideal has infinite residue field. Theorem 2.2 implies that there exists an infinite set $S^*$ in $R$ such that $x - y$ is a unit in $R_E$ for all $x \neq y$ in $S^*$. Choose $m \in \bigcap_i M_i \setminus \bigcup_i M_i \ (i > n)$ and set $S = \{ms | s \in S^*\}$. Then $S$ is an infinite set in $R$ such that $S \subset \bigcap_i M_i$ and $x - y$ is a unit in $R_E$ for all $x \neq y$ in $S$. Let $x_1, x_2, x_3, \ldots, x_i, \ldots$ be a denumerable subset of $S$. Since $R_B$ is a Bezout ring, we have $AR_B = aR_B$ for some $a \in A$. Consider the expressions

\[ b_i = a + \sum_{j=1}^{m} a_j x_i^j \quad \text{for } i = 1, 2, 3, \ldots, \infty \]

Then there must be $r_1 + 1$ values of the index $i$ such that $b_i$ belongs to some of the $A_j$, say $b_1, \ldots, b_{m+1}$ belong to $A_1$. We claim that $AR_{M_i} = A_1 R_{M_i}$ for $i = 1, \ldots, r$, and therefore $A = A_1$. We deal first with the case $i > n$. Solving the system of equations $b_i = a + \sum_{j=1}^{m} a_j x_i^j$ for the $a_j$ by Cramer's rule, we see that $a_d \in A_1$, where $d = \Pi(x_i - x_j) \ (i > j)$ since it is a Vandermonde determinant. Consequently $a_i R_E \subset A_1 R_E$ for $j = 1, \ldots, m$ and $AR_E = A_1 R_E$; hence $AR_{M_i} = A_1 R_{M_i}$ for $i > n$. On the other hand, $b_1 R_B = aR_B$ since $a_i R_B \subset AR_B = aR_B$ for $i = 1, \ldots, m$ implies "$a$ divides $a_i$" in $R_B$ and hence $b_1 R_B = a(1 + j)R_B$ where $j$ is in the Jacobson radical of $R_B$. Hence $aR_B \subset A_1 R_B \subset AR_B = aR_B$, $AR_B = A_1 R_B$, and $AR_{M_i} = A_1 R_{M_i}$ for $i = 1, \ldots, n$, completing the proof.

Theorem 2.6. The following statements are equivalent.

(a) $R$ is a $u$-ring.
(b) $R_p$ is a $u$-ring for every maximal ideal $P$ of $R$.
(c) For each maximal ideal $P$ of $R$, either the residue field $R/P$ is infinite or the quotient ring $R/P$ is a Bezout ring.

Proof. (a) $\Rightarrow$ (b) by Proposition 1.3, (b) $\Rightarrow$ (c) by Corollary 1.9, and (c) $\Rightarrow$ (a) by Proposition 2.1 and Theorem 2.5.

3. Some applications and examples. Applying Proposition 1.7, it is clear that a unitary overring of an infinite field is a um-ring, and that the ring $R(x)$ is a um-ring for any ring $R$ [N, p. 18], [G, p. 410]. The following results can be established by standard techniques using Theorems 2.3 and 2.6. A finite direct sum of rings is a um-ring (u-ring) if and only if each summand is; and if any one of $R, R[x], R[[x]]$ is a um-ring, so are the other two. Let $D$ be an integral domain with quotient field $K$. If $D$ is a $u$-ring and $D \subset R \subset K$, then $R$ is a $u$-domain; if $D/P$ is finite for all maximal ideals $P$ of $D$, then $D$ is a $u$-domain if and only if $D$ is a Prufer domain. If $F$ is a finite algebraic extension field of the rational numbers and $F \supset R \supset Z$, then $R$ is a $u$-ring if and only if $R$ is integrally closed. Let $J$ be the integral closure of $D$ in an algebraic extension field $L$ of $K$. If $D$ is a $u$-ring, then $J$ is a $u$-ring; if $J$ is a $u$-ring, $[L : K]$ is finite, and $D$ is integrally closed, then $D$ is a $u$-ring.
Of course, a finite ring $R$ is a $u$-ring if and only if $R$ is a principal ideal ring.

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