FINITE UNIONS OF IDEALS AND MODULES

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ABSTRACT. We say that a commutative ring $R$ is a $u$-ring provided $R$ has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a $um$-ring is a ring $R$ with the property that an $R$-module which is equal to a finite union of submodules must be equal to one of them. The primary purpose of this paper is to characterize $u$-rings and $um$-rings. We show that $R$ is a $um$-ring if and only if for each maximal ideal $P$ of $R$ and $R$ is a $u$-ring if and only if for each maximal ideal $P$ of $R$ either the residue field $R/P$ is infinite or the quotient ring $R_P$ is a Bézout ring.

Introduction. All rings considered are commutative with identity $1 \neq 0$, and $R$ denotes a ring with total quotient ring $T(R)$. For a multiplicative system $S$ of $R$ and a unitary $R$-module $M$, we use standard conventions concerning properties and notation for the quotient ring $R_S$ and the quotient module $M_S$ (e.g. see [G, p. 52], [N, p. 14], [K, p. 22], [ZS, p. 221]); in particular, $AR_S$ will denote the ideal in $R_S$ associated with the ideal $A$ of $R$ by the canonical map from $R$ to $R_S$. The "running indices" will usually be dropped from finite intersections, unions, and sums; thus $\bigcup A_i$, $\bigcap A_i$, and $\sum A_i$ will uniformly mean that $i$ ranges from $i = 1$ to $i = n$. An ideal $A$ of $R$ is called a $u$-ideal provided $A \subset \bigcup A_i$ implies $A \subset A_i$ for some $i$, where $A_1, \ldots, A_n$ are ideals of $R$.

It is well known that an ideal of $R$ contained in a finite union of prime ideals of $R$ must be contained in one of those primes [K, p. 55], [M], but it does not seem to have been generally observed that this property holds in certain rings (e.g. Dedekind domains) even if none of the ideals involved are prime. In [M], McCoy shows that $A \subset \bigcup A_i$ implies $A^k \subset \bigcap A_i$ for some positive integer $k$ in a general commutative ring, provided $A$ is not contained in the union of any $n - 1$ of the $A_i$; in addition he proves an analogous result for subgroups of a group $G$.

1. Preliminary results. In this section we show that in considering $u$-rings we can replace $A \subset \bigcup A_i$ by $A = \bigcup A_i$, thus motivating the definition of $um$-rings; moreover, only finitely generated $A$ need be considered. In addition, we show that every invertible ideal is a $u$-ideal.
The proofs of the following two propositions are easy and we omit them.

**Proposition 1.1.** The following conditions are equivalent, for $A, A_1, \ldots, A_n$ ideals of $R$.

1. Each ideal of $R$ is a u-ideal.
2. Each finitely generated ideal of $R$ is a u-ideal.
3. $A = \bigcup A_i$ finitely generated $\Rightarrow A = A_i$ for some $i$.
4. $A = \bigcup A_i \Rightarrow A = A_i$ for some $i$.

**Corollary 1.2.** Every Bézout ring (i.e., a ring in which every finitely generated ideal is principal) is a u-ring.

**Proposition 1.3.** If $R$ is a u-ring (um-ring), then every homomorphic image and every quotient ring $R_\alpha$ of $R$ is a u-ring (um-ring).

It is clear that a um-ring is a u-ring, but the converse is false (e.g., the ring of integers). It is easy to check that if $M = \bigcup M_i$ implies $M = M_i$ for some $i$, where $M$ is a finitely generated $R$-module and $M_1, \ldots, M_n$ are submodules of $M$, then $R$ is a um-ring.

**Proposition 1.4.** If (0) is the annihilator of the finitely generated ideal $A$ of $R$ and $B_i \neq R$ is an ideal of $R$ for $i = 1, \ldots, n$, then $A \not\subseteq \bigcup AB_i$.

**Proof.** We use induction on $n$. Consider $n = 1$ and suppose $A = AB_1$. There exists $b \in B_1$ such that $a = ab$ for all $a \in A$ [ZS, p. 215], implying $B_1 = R$, a contradiction. Suppose the proposition holds for $n - 1 > 1$ and consider two cases.

**Case 1.** Among $B_1, \ldots, B_n$ there are two ideals, say $B_1$ and $B_2$, such that $B_1 + B_2 \neq R$. Since $A \not\subseteq A(B_1 + B_2) \cup AB_3 \cup \ldots \cup AB_n$, it follows that $A \not\subseteq \bigcup AB_i$, completing Case 1. In Case 2, suppose $B_i + B_j = R$ for all $i \neq j$. Let $C_i = \bigcup B_i$ for $i = 1, \ldots, n$. Then $B_i + C_i = R$ [ZS, p. 177] and $AB_i + AC_i = A$ for $i = 1, \ldots, n$. If $AC_i \subset AB_i$ for some $i$, then $A = AB_i + AC_i = AB_i$, which is impossible by the case $n = 1$. Hence there exists $a_i \in AC_i \setminus AB_i$ for $i = 1, \ldots, n$. Set $a = \Sigma a_i$ and note that $a \in A \setminus AB_i$ for $i = 1, \ldots, n$, completing the proof.

By a fractional ideal $F$ of $R$ we understand an $R$-module $F \subset T(R)$ for which there exists a regular element (i.e., not a zero divisor) $r$ of $R$ such that $rF \subset R$.

**Theorem 1.5.** Every invertible ideal of $R$ is a u-ideal.

**Proof.** If $A$ is an invertible ideal and $A_1, \ldots, A_n$ are ideals of $R$ such that $A \subset \bigcup A_i$, then $A = \bigcup C_i$ where $C_i = A \cap A_i$. Suppose $A \neq C_i$ for $i = 1, \ldots, n$. Then $C_i = AB_i$ for each $i$, where $R \neq B_i = C_iA^{-1}$ is an ideal of $R$. Since $A$ is invertible, the annihilator of $A$ is (0) and $A$ is finitely generated [ZS, p. 272]. Proposition 1.4 implies $A \not\subseteq \bigcup AB_i = \bigcup C_i$, a contradiction.
Corollary 1.6. Every Prüfer domain is a u-domain.

Griffin [MG] defines a Prüfer ring as a ring in which every finitely generated regular ideal is invertible; this type of Prüfer ring need not be a u-ring as can be seen by taking $R = \mathbb{Z}[x]/(2, x)^2$, where $\mathbb{Z}[x]$ is the polynomial ring over the integers. Butts and Smith [BS] called $R$ a Prüfer ring provided the ideals of $R$ are linearly ordered by inclusion for every proper prime ideal $P$ of $R$; it follows from the characterization given in §2 that Prüfer rings of this type are u-rings.

Proposition 1.7. If $R$ contains an infinite set $S$ such that $x - y$ is a unit of $R$ for all $x \neq y$ in $S$, then $R$ is a um-ring.

Proof. Suppose $M_1, \ldots, M_n$ are submodules of an $R$-module $M$ such that $M = \bigcup M_i$ and $M \neq M_i$ for each $i$. It is no restriction to assume for each $i$ that $M_i \neq \bigcup M_j$ ($j \neq i$), so let $m_i \in M_i$ and $m_i \notin \bigcup M_j$ ($j \neq i$) for $i = 1, 2$ and consider the set $E = \{m_1 + xm_2 | x \in S\}$. There exist $x$ and $y$ in $S$ such that $x \neq y$ and both $m_1 + xm_2$ and $m_1 + ym_2$ belong to the same $M_i$ for some $i \neq 2$, implying $m_2 \in M_i$, a contradiction.

Theorem 1.8. If there exists a maximal ideal $P$ of $R$ such that $F = R/P$ is finite with $n - 1$ elements, then for an $R$-module $M$ (ideal $M_i$) such that the vector space $V = M/PM$ is not 1-dimensional over $F$ there exist $n$ submodules (ideals of $R$) $M_i \subset M$ such that $M \neq M_i$ for $i = 1, \ldots, n$ and $M = \bigcup M_i$.

Proof. Let $B$ be a basis for $V$ over $F$ and let $b_1 \neq b_2$ be two elements of $B$. Consider the following subsets of $V$: $E_1 = B \setminus \{b_1\}$, $E_2 = B \setminus \{b_2\}$, and $E_{2+i} = (B \setminus \{b_1, b_2\}) \cup \{b_1 + x_i b_2\}$ where $x_i$ ($i = 1, \ldots, q - 1$) ranges over the nonzero elements of $F$. Let $S_j$ be the subspace of $V$ generated by $E_j$ for $j = 1, \ldots, n$. A routine argument shows that $V = \bigcup S_i$ and $S_i \neq V$ for $i = 1, \ldots, n$. Hence, if $f : M \to V$ is the canonical homomorphism and $M_i = f^{-1}(S_i)$ for $i = 1, \ldots, n$, $M = \bigcup M_i$ and $M \neq M_i$ for each $i$.

Corollary 1.9. Let $R$ be a quasi-local ring with maximal ideal $P$.

(i) If $R$ is a um-ring, then $R/P$ is infinite.

(ii) If $R$ is a u-ring, then either $R/P$ is infinite or $R$ is a Bézout ring.

Proof. If $M$ is finitely generated and not principal, then so is $M/PM$ as a vector space over $R/P$ [N, p. 13]; moreover, $M = R \oplus R$ is such a module.

2. Characterization of u-rings and um-rings. We first reduce the general case to the case in which $R$ has only finitely many maximal ideals, and then deal with that situation.

Proposition 2.1. A ring $R$ is a um-ring (u-ring) if and only if $R_S$ is a um-ring (u-ring) for each multiplicative system $S$ of $R$ which is the complement of a finite union of maximal ideals of $R$. 
Proof. If \( R \) is a \( \text{um-ring} \) (\( \text{u-ring} \)), then so is \( R_S \) by Theorem 1.3. Now, consider the converse and suppose \( M = \bigcup M_i \), where \( M_i \) is a proper sub-module of the \( R \)-module \( M \) (ideal \( M \) of \( R \)) for \( i = 1, \ldots, n \). Let \( m_i \in M \setminus M_i \) and let \( P_i \) be a maximal ideal of \( R \) containing the ideal \( A_i = [M_i : m_i]_R \) for \( i = 1, \ldots, n \). If \( S \) denotes the complement of \( \bigcup P_i \) in \( R \), then \( M_S = \bigcup (M_i)_S \) is clear and \( M_S \not\subseteq (M_i)_S \) since \( m_i \not\in (M_i)_S \).

Theorem 2.2. If \( R \) has only finitely many maximal ideals, say \( M_1, \ldots, M_n \), then the following statements are equivalent.
(a) \( R \) is a \( \text{um-ring} \).
(b) \( R_P \) is a \( \text{um-ring} \) for every maximal ideal \( P \) of \( R \).
(c) \( R/P \) is infinite for every maximal ideal \( P \) of \( R \).
(d) There exists an infinite set \( S \) in \( R \) such that \( x - y \) is a unit in \( R \) for all \( x \neq y \) in \( S \).

Proof. (a) \( \Rightarrow \) (b) by Proposition 1.3, (b) \( \Rightarrow \) (c) by Corollary 1.9, and (d) \( \Rightarrow \) (a) by Proposition 1.7. Consider (c) \( \Rightarrow \) (d). Let \( S_i \) be a complete set of representatives for the nonzero elements of \( R/M_i \), and let \( \{s_{ij} : j = 1, 2, \ldots, \infty\} \) be a denumerable subset of \( S_i \) for \( i = 1, \ldots, n \). For each \( j \), there exists \( x_j \) in \( R \) such that \( x_j \equiv s_{ij} \) (mod \( M_i \)) for \( i = 1, \ldots, n \) [ZS, p. 177]. Denote by \( S \) the set of all \( x_j \) thus obtained and observe that for \( x_r \neq x_s \) in \( S \) we have \( x_r - x_s \notin \bigcup M_i \) and \( x_r - x_s \) is a unit of \( R \).

Theorem 2.3. The following statements are equivalent.
(a) \( R \) is a \( \text{um-ring} \).
(b) \( R_P \) is a \( \text{um-ring} \) for every maximal ideal \( P \) of \( R \).
(c) \( R/P \) is infinite for every maximal ideal \( P \) of \( R \).

Proof. This follows directly from Propositions 1.3, 2.1, and Theorem 2.2.

Lemma 2.4. If \( R \) has only finitely many maximal ideals and \( R_M \) is a \( \text{Bézout ring} \) for each maximal ideal \( M \) of \( R \), then \( R \) is a \( \text{Bézout ring} \).

Proof. Let \( M_1, \ldots, M_n \) be the maximal ideals of \( R \) and for each \( i = 1, \ldots, n \), choose \( m_i \in M_i \) for \( j \neq i \) and \( m_i \not\in M_i \). For a finitely generated ideal \( A \) of \( R \), there exists \( a_i \in A \) such that \( AR_{M_i} = a_i R_{M_i} \) for \( i = 1, \ldots, n \). If \( A = \sum a_i m_i \), then \( AR_{M_i} = a_i R_{M_i} \) for \( i = 1, \ldots, n \) since \( a_i R_{M_i} \subseteq AR_{M_i} = a_i R_{M_i} \) implies \( "a_i \) divides \( a_j" \) in \( R_{M_i} \), and \( aR_{M_i} = a_i u_i R_{M_i} \) where \( u_i \) is a unit in \( R_{M_i} \). Consequently \( A = aR \).

Theorem 2.5. If \( R \) has only finitely many maximal ideals, say \( M_1, \ldots, M_r \), and \( R_{M_i} \) is a \( \text{Bézout ring} \) for \( i = 1, \ldots, n \) \( (1 < n \leq r) \) while \( R/M_i \) is infinite for \( i > n \), then \( R \) is a \( \text{u-ring} \).

Proof. We can assume \( n = r \) since \( x \rightarrow y \) implies \( R \) is a \( \text{Bézout ring} \) by Lemma 2.4 and therefore a \( \text{u-ring} \). Suppose \( A = (a_1, \ldots, a_m) \) and \( A_1, \ldots, A_s \)
are ideals in $R$ with $A = A_1 \cup \ldots \cup A_s$. For $B = R \setminus \bigcup M_i$ and $E = R \setminus \bigcup M_i (i > n)$, $R_B$ is a Bezout ring by Lemma 2.4 and $R_E$ has the property that each maximal ideal has infinite residue field. Theorem 2.2 implies that there exists an infinite set $S^*$ in $R$ such that $x - y$ is a unit in $R_E$ for all $x \neq y$ in $S^*$. Choose $m \in \bigcap M_i \setminus \bigcup M_i (i > n)$ and set $S = \{ms | s \in S^*\}$. Then $S$ is an infinite set in $R$ such that $S \subset \bigcap M_i$ and $x - y$ is a unit in $R_E$ for all $x \neq y$ in $S$. Let $x_1, x_2, x_3, \ldots, x_i, \ldots$ be a denumerable subset of $S$. Since $R_B$ is a Bezout ring, we have $AR_B = aR_B$ for some $a \in A$. Consider the expressions $b_i = a + \sum_{j=1}^{m} a_j x_i^j$ for $i = 1, 2, 3, \ldots, \infty$. There must be $m + 1$ values of the index $i$ such that $b_i$ belongs to some one of the $A_j$, say $b_1, \ldots, b_{m+1}$ belong to $A_1$. We claim that $AR_{M_i}^{R} = A_1 R_{M_i}^{R}$ for $i = 1, \ldots, r$, and therefore $A = A_1$. We deal first with the case $i > n$. Solving the system of equations $b_i = a + \sum_{j=1}^{m} a_j x_i^j$ for $1 \leq i \leq m + 1$ for the $a_j$ by Cramer's rule, we see that $a_i d \in A_1$ where $d = \prod (x_i - x_j)$ (i > j) since it is a Vandermonde determinant. Consequently $a_i R_E \subset A_1 R_E$ for $j = 1, \ldots, m$ and $AR_E = A_1 R_E$; hence $AR_{M_i}^{R} = A_1 R_{M_i}^{R}$ for $i = 1, \ldots, n$. On the other hand, $b_i R_B = a R_B$ since $a_i R_B \subset AR_B = a R_B$ for $i = 1, \ldots, m$ implies "$a$ divides $a_i$" in $R_B$ and hence $b_i R_B = a(1 + j) R_B$ where $j$ is in the Jacobson radical of $R_B$. Hence $a R_B \subset A_1 R_B \subset AR_B = a R_B$, $AR_B = A_1 R_B$, and $AR_{M_i}^{R} = A_1 R_{M_i}^{R}$ for $i = 1, \ldots, n$, completing the proof.

**Theorem 2.6.** The following statements are equivalent.

(a) $R$ is a $u$-ring.
(b) $R_P$ is a $u$-ring for every maximal ideal $P$ of $R$.
(c) For each maximal ideal $P$ of $R$, either the residue field $R/P$ is infinite or the quotient ring $R/P$ is a Bezout ring.

**Proof.** (a) $\Rightarrow$ (b) by Proposition 1.3, (b) $\Rightarrow$ (c) by Corollary 1.9, and (c) $\Rightarrow$ (a) by Proposition 2.1 and Theorem 2.5.

3. Some applications and examples. Applying Proposition 1.7, it is clear that a unitary overring of an infinite field is a $um$-ring, and that the ring $R(x)$ is a $um$-ring for any ring $R$ [N, p. 18], [G, p. 410]. The following results can be established by standard techniques using Theorems 2.3 and 2.6. A finite direct sum of rings is a $um$-ring ($u$-ring) if and only if each summand is; and if any one of $R, R[x], R[[x]]$ is a $um$-ring, so are the other two. Let $D$ be an integral domain with quotient field $K$. If $D$ is a $u$-ring and $D \subset R \subset K$, then $R$ is a $u$-domain; if $D/P$ is finite for all maximal ideals $P$ of $D$, then $D$ is a $u$-domain if and only if $D$ is a Prüfer domain. If $F$ is a finite algebraic extension field of the rational numbers and $F \subset R \subset Z$, then $R$ is a $u$-ring if and only if $R$ is integrally closed. Let $J$ be the integral closure of $D$ in an algebraic extension field $K$ of $K$. If $D$ is a $u$-ring, then $J$ is a $u$-ring; if $J$ is a $u$-ring, $[L: K]$ is finite, and $D$ is integrally closed, then $D$ is a $u$-ring.
Of course, a finite ring $R$ is a $u$-ring if and only if $R$ is a principal ideal ring.

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