WEAKLY COMPLETELY CONTINUOUS ELEMENTS OF C*-ALGEBRAS

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ABSTRACT. For a C*-algebra $A$ and $u \in A$, the equivalence of the following three statements is proved: (i) the map $x \mapsto uxu$ is a compact operator on $A$, (ii) (resp. (iii)) the map $x \mapsto ux$ (resp. $x \mapsto xu$) is a weakly compact operator on $A$. The canonical image of a dual C*-algebra $A$ in its bidual $A^{**}$ is characterized as the set of the weakly completely continuous elements of $A^{**}$.

1. Introduction. Let $E$ be a Banach space and $L(E)$ the Banach algebra of bounded linear operators on $E$. K. Vala has proved in [15] that $T \in L(E)$ is a compact operator on $E$ if and only if the map $X \mapsto TXT$ is a compact operator on $L(E)$. Motivated by this phenomenon Vala defined in [16] the element $u$ of an arbitrary Banach algebra $A$ to be compact, if the map $x \mapsto uxu$ is a compact operator on $A$. Subsequent investigations (see [1], [18], [19]) have further indicated that this definition yields indeed a natural extension of the notion of a compact operator.

If $H$ is a Hilbert space, the following nonspatial characterization of the compact operators on $H$ is also available: T. Ogasawara proved in [10] that $T \in L(H)$ is a compact operator if and only if the map $X \mapsto TX$ is a weakly compact operator [6, p. 482] on $L(H)$. In the context of C*-algebras this result suggests another generalization of the concept of a compact operator. For any Banach algebra $A$, $u \in A$ is called a left (resp. right) weakly completely continuous—abbreviated l.w.c.c. (resp. r.w.c.c.)—element of $A$, if the map $x \mapsto ux$ (resp. $x \mapsto xu$) is a weakly compact operator on $A$. It follows from Corollary 6 in [6, p. 484] that the l.w.c.c. (resp. r.w.c.c.) elements of $A$ form a closed two-sided ideal. In the case of a C*-algebra these ideals are thus selfadjoint [5, p. 17], and so (as noted by Ogasawara in [10, p. 362]) if $u \in A$ is l.w.c.c. it is also r.w.c.c. (the operator $x \mapsto (u^*x^*)^* = xu$ is weakly compact), and conversely. Therefore we shall simply call the l.w.c.c. (resp. r.w.c.c.) elements of a C*-algebra weakly completely continuous (w.c.c.).

The main result of this paper (Theorem 3.1) states that an element of a C*-algebra is compact if and only if it is w.c.c., i.e., the two generalizations of a compact operator are in fact the same. The first half of our proof is based on the theorem of Ogasawara mentioned above, but in §2 we give this
result a short new proof (Corollary 2). We are grateful to the referee for simplifying the second half of the proof of Theorem 3.1.

Let $A$ be a $C^*$-algebra. We shall regard its second conjugate space $A^{**}$ as a $C^*$-algebra by identifying it with the enveloping von Neumann algebra of $A$ [5, p. 237]. In [3, p. 869] it is proved that this algebra structure of $A^{**}$ also arises from either one of the two Arens multiplications in $A^{**}$. If $x \in A$, we let $\tilde{x}$ denote the canonical image of $x$ in $A^{**}$, and write $\check{A} = \{\tilde{x} \mid x \in A\}$.

2. Dual $C^*$-algebras. It is well known (see [14, p. 533], [8], [2, p. 255]) that a $C^*$-algebra $A$ is dual in the sense of Kaplansky (see e.g. [7]) if and only if $\check{A}$ is an ideal (two-sided by selfadjointness) of $A^{**}$. Several characterizations of dual $C^*$-algebras are listed in [5, 4.7.20, p. 99]. We note in passing that one further criterion, given in [9, p. 88] (see also [17, p. 538]), follows at once from Theorem 6 in [11, p. 21] (i.e. (v) in [5, p. 99]) and Gantmacher's theorem [6, p. 485].

Theorem 2.1. Let $A$ be a dual $C^*$-algebra and $u \in A^{**}$. Then $u \in \check{A}$ if and only if $u$ is a w.c.c. element of $A^{**}$.

Proof. Suppose first that $u = \tilde{a}$ for some $a \in A$. The definition of the first Arens product in $A^{**}$ (see e.g. [3, p. 848]) shows immediately that the map $x \mapsto ux$ on $A^{**}$ is the second transpose $L_{u}^{**}$ of $L_{a}^{**}$ defined by $L_{a}^{**}x = ax$, $x \in A$. As $\check{A}$ is an ideal in $A^{**}$, $L_{a}^{**}x = ux \in \check{A}$ for all $x \in A^{**}$. Thus $L_{a}^{**}$ is weakly compact [6, p. 482], and so is $L_{u}^{**} : A^{**} \to A^{**}$ by Gantmacher's theorem [6, p. 485], i.e. $u$ is w.c.c. Suppose, conversely, that $u$ is w.c.c. Since $\check{A}$ is an ideal in $A^{**}$, $u\check{A} \subset \check{A}$, so the restriction $L = L_{u}^{**} \mid \check{A}$, where $L_{u}^{**}x = ux$, $x \in A^{**}$, may be regarded as an operator from $A$ into itself. We have $L_{u}^{**} = L_{u}^{**}$, since both operators are weak* continuous (see [6, p. 478], [3, pp. 848, 869]) and agree on the weak* dense subspace $\check{A}$ of $A^{**}$. Another application of Gantmacher's theorem shows that $L$ is weakly compact so that $L_{u}^{**}(A^{**}) \subset \check{A}$ [6, p. 482], i.e. $uA^{**} \subset \check{A}$. In particular, for the identity $1$ of $A^{**}$ we have $u = u1 \in \check{A}$.

Corollary 1. Let $A$ and $B$ be dual $C^*$-algebras. Any topological algebra isomorphism from $A^{**}$ onto $B^{**}$ maps $\check{A}$ onto $\check{B}$. In particular, if $A^{**}$ and $B^{**}$ are *-isomorphic, then so are $A$ and $B$.

Remark. Corollary 1 becomes false, if the word "dual" is omitted. As an example one may consider the $C^*$-algebra $c$ of all convergent sequences of complex numbers, and its sub-$C^*$-algebra $c_{0}$ consisting of the sequences converging to zero. It is well known that the second conjugate space of both $c$ and $c_{0}$ is $l^{\infty}$, the space of bounded sequences, but $c$ is not isometrically isomorphic to $c_{0}$.

Corollary 2 (Ogasawara [10, Theorem 4, p. 362]). Let $H$ be a Hilbert
space and $T \in L(H)$. Then $T$ is a compact operator on $H$ if and only if $T$ is a w.c.c. element of $L(H)$.

**Proof.** Let $C(H)$ denote the ideal of $L(H)$ consisting of all compact operators on $H$. Since $L(H)$ may be identified with $C(H)^{**}$ in such a way that the canonical embedding of $C(H)$ into $C(H)^{**}$ corresponds to the inclusion map of $C(H)$ into $L(H)$ (see e.g. [5, p. 236] or [13, p. 47]), the corollary is an immediate consequence of the theorem.

3. The equivalence of compactness and weak complete continuity for elements of $C^*$-algebras. The proof below that (ii) implies (i) is due to the referee. It is considerably shorter than our original argument.

**Theorem 3.1.** Let $A$ be a $C^*$-algebra and $u \in A$. The following three conditions are equivalent:

(i) the map $x \mapsto u x u$ is a compact operator on $A$,

(ii) (resp. (iii)) the map $x \mapsto u x$ (resp. $x \mapsto x u$) is a weakly compact operator on $A$.

**Proof.** It was noted in the introduction that (ii) and (iii) are equivalent. Assume (i). There is an isometric $*$-representation $\pi$ of $A$ on a Hilbert space $H$ such that $\pi(u)$ is a compact operator on $H$ [19]. By Corollary 2 in §2, the operators $X \mapsto \pi(u)X$ and $X \mapsto X\pi(u)$ on $L(H)$ are weakly compact. Since $\pi(A)$ is $\sigma(L(H), L(H)^*)$-closed and the relative $\sigma(L(H), L(H)^*)$-topology on $\pi(A)$ agrees with $\sigma(\pi(A), \pi(A)^*)$, it follows that $x \mapsto u x$ and $x \mapsto x u$ are weakly compact operators on $A$. Assume now (ii). As the ideal $W$ of the w.c.c. elements is selfadjoint, it is a sub-$C^*$-algebra of $A$. Since each element of $W$ is w.c.c., $W$ is a dual $C^*$-algebra by Theorem 6 in [11, p. 21]. As $W$ has an approximate identity [5, p. 15], Cohen's factorization theorem [4, Theorem 1] shows that $u = v w$ for some $v, w \in W$. Thus the operator $x \mapsto u x u$ on $A$ may be written as $T_2 T_1 T_3 u$, where $T_1 x = x v$, $x \in A$, $T_2 y = w y w$, $y \in W$, and $T_3 z = z v$, $z \in W$. But $T_2 : W \to W$ is a compact operator (see e.g. [1, Corollary 8.3]), and so (i) holds.

**Note.** The compact elements of $A$ form a closed two-sided ideal [18, Theorem 3.10]. This ideal is by Corollary 8.3 in [1] a dual $C^*$-algebra, whose every element is thus w.c.c. by [11, Theorem 6]. It is therefore clear that the technique used in the second half of the above proof would give an alternate approach to the first half of the proof, too.

**Corollary 1.** The $C^*$-algebra $A$ is dual if and only if its canonical image in $A^{**}$ coincides with the closure of the socle of $A^{**}$.

**Proof.** The socle of a Banach algebra is discussed in [12, p. 46]. Since the socle, if it exists, is a two-sided ideal, the condition is sufficient for $A$ to be dual. Suppose now that $A$ is dual. Theorems 2.1 and 3.1 show that
\( \tilde{A} \) coincides with the set of the compact elements of \( A^{**} \). But this set is the closure of the socle of \( A^{**} \) by Theorems 3.10 and 5.1 in [18].

Of course, Theorem 3.1 transfers all known facts about compact elements of \( C^* \)-algebras (for example, the representation theorem of [19]) to the context of w.c.c. elements. In particular, we obtain the following generalization of Ogasawara's theorem (Corollary 2 in §2):

**Corollary 2.** Let \( H \) be a Hilbert space and \( A \) an irreducible sub-\( C^* \)-algebra of \( L(H) \). Then \( T \in A \) is a w.c.c. element of \( A \) if and only if \( T \) is a compact operator on \( H \).

**Proof.** We only need to show that \( T \) is a compact operator, if it is a w.c.c. element of \( A \). This follows from the above theorem and Corollary 2 in [18, p. 15]. (Note that for \( C^* \)-algebras strict and topological irreducibility are equivalent [5, p. 45].)

**REFERENCES**