EXTENDING CONTINUOUS FUNCTIONS IN ZERO-DIMENSIONAL SPACES

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ABSTRACT. Suppose that $X$ is a completely regular, zero-dimensional space and that a dense subset $S$ of $X$ is not $C^*$-embedded in $X$; does there then exist a two-valued function in $C^*(S)$ with no continuous extension to $X$? The answer is negative even if $X$ is a compact space. The question was raised by N. J. Fine and L. Gillman in Extension of continuous functions in $\beta N$, Bull. Amer. Math. Soc. 66 (1960), 376–381.

This paper answers a question raised by N. J. Fine and L. Gillman in [1]. Suppose that $X$ is a completely regular, zero-dimensional space and that a dense subset $S$ of $X$ is not $C^*$-embedded in $X$; does there then exist a two-valued function in $C^*(S)$ with no continuous extension to $X$? Theorem 1 establishes that the answer is negative.

I. First, I will give some background material, all of which can be found in [2].

All topological spaces are assumed to be completely regular.

The set of all bounded, continuous, real-valued functions on $X$ will be denoted by $C(X)$. A subspace $S$ of $X$ is $C^*$-embedded in $X$ iff every function in $C^*(S)$ can be extended to a function in $C^*(X)$. The Stone-$\check{C}$ech compactification of $X$ is denoted as $\beta X$; that is $\beta X$ is the compactification of $X$ in which $X$ is $C^*$-embedded.

A space $X$ is zero-dimensional if any two completely separated sets in $X$ are contained in complementary open-and-closed sets of $X$. A space $X$ is zero-dimensional if and only if $X$ is zero-dimensional.

The space of countable ordinals with the order topology will be denoted by $W$.

II. Theorem 1. There exists a zero-dimensional space $X$ having a dense subset $S$ such that $S$ is not $C^*$-embedded in $X$, but every two-valued function in $C^*(S)$ has a continuous extension to $X$.

Proof. Let $I = [0, 1]$ with the usual topology. For each $\alpha \in W$, select $I_\alpha \subset I$ such that $I_\alpha$ is dense in $I$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$, and such that $\bigcup_{\alpha \in W} I_\alpha = I$. 

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Let \( S_\alpha = \{ (x, \alpha) : x \in \bigcup_{\beta > \alpha} I_\beta \} \) and \( S = \bigcup_{\alpha \in \mathbb{W}} S_\alpha \). Then \( S \subseteq I \times \mathbb{W} \).

Topologize \( S \) using the relative topology from \( I \times \mathbb{W} \).

Note that the collection of all neighborhoods of \( (x, \alpha) \) of the form \( \{(r, y) : y < r < z \text{ and } \delta < y \leq \alpha \} \) where \( y < x < z \) and \( y \) and \( z \) belong to \( \bigcup_{\beta > \alpha} I_\beta \) is a basis of open-closed neighborhoods of \( (x, \alpha) \) since \( \bigcup_{\beta > \alpha} I_\beta \) is dense in \( I \).

Let \( X = S \cup \{ 2 \} \). We define a topology on \( X \) as follows. \( S \) will be an open subspace of \( X \). A neighborhood of \( 2 \) is any set \( U \) containing \( 2 \) such that \( 2 \in U \) and there is a \( \beta \in \mathbb{W} \) such that \( \{(x, \alpha) : \alpha > \beta \} \subseteq U \).

Since every neighborhood of \( 2 \) intersects \( X \), \( S \) is dense in \( X \). Also, \( S \) is completely regular since \( S \subseteq I \times \mathbb{W} \) where both \( I \) and \( \mathbb{W} \) are completely regular.

A consequence of the following proof that \( X \) is zero-dimensional is that \( X \) is normal. So \( X \) is, clearly, completely regular.

To show that \( X \) is zero-dimensional, I will show that any two disjoint closed sets in \( X \) are contained in complementary open-closed sets. First, consider the case where \( A \) and \( B \) are disjoint closed sets in \( X \) such that \( A \cup B \subseteq C = \bigcup_{\alpha \leq \gamma} S_\alpha \) for some \( \gamma \in \mathbb{W} \). Consider \( \alpha_0 \leq \gamma \). For each point \( (x, \alpha_0) \in S_{\alpha_0} \) pick \( U(x) \) a basic open-closed neighborhood of \( (x, \alpha_0) \) such that either \( U(x) \cap A \) or \( U(x) \cap B \) is empty. Identifying \( S_{\alpha_0} \) with \( I - \bigcup_{\beta > \alpha_0} I_\beta \), \( S_{\alpha_0} \) is second countable, so a countable collection \( \{ U(x) : n \in \mathbb{N} \} \) covers \( S_{\alpha_0} \). Now, since \( \gamma \in \mathbb{W} \) there is a countable collection, say \( \{ V_n : n \in \mathbb{N} \} \) of open-closed sets, covering \( C \) with the property that for each \( n \), either \( V_n \cap A \) or \( V_n \cap B \) is empty. Define \( W_n = V_n - \bigcup_{i < n} V_i \). Then \( \bigcup_{n \in \mathbb{N}} W_n \) is a collection of disjoint open-closed sets which covers \( C \) and either \( W_n \cap A \) or \( W_n \cap B \) is empty. Let \( 0 = \bigcup \{ W_k : \bigcap A = \emptyset \} \); then \( C - 0 = \bigcup \{ W_k : \bigcap A \neq \emptyset \} \). So \( 0 \) and \( C - 0 \) are complementary open-closed sets in \( C \) and \( A \subseteq C - 0 \) and \( B \subseteq 0 \). Since \( X - C \) is open and closed in \( X \), \( 0 \cup X - C \) and \( C - 0 \) are complementary open-closed sets in \( X \).

Now, suppose \( A \) and \( B \) are disjoint closed sets in \( X \) and \( 2 \in A \). Then there exists a \( \beta \in \mathbb{W} \) such that \( B \subseteq D = \bigcup_{\alpha \leq \beta} S_\alpha \). Since \( D \) is closed in \( X \), \( A \cap D \) is closed in \( X \). By the above argument there exist complementary open-closed sets \( H \) and \( K \) in \( D \) such that \( B \subseteq H \) and \( A \cap D \subseteq K \). Then \( H \) and \( K \cup X - D \) are complementary open-closed sets in \( X \) such that \( B \subseteq H \) and \( A \subseteq H \cup X - D \). So \( X \) is zero-dimensional.

To show that \( S \) is not \( C^* \)-embedded in \( X \), define \( F : S \to I \) by \( F((x, \alpha)) = x \). Obviously \( F \) is continuous.

However, \( F \) cannot be extended continuously to \( 2 \), since \( F \) assumes all values in every neighborhood of \( 2 \).

Every two-valued continuous function on \( S \) can be extended continuously to \( X \). Let \( f \) be a two-valued continuous function on \( S \) with range \( 10, 11 \). For
each \( x \in I \), there exists an \( \alpha_x \in W \) such that \( f \) is constant on \( \{ (x, \beta) : \beta \geq \alpha_x \} \), since for fixed \( x \), the set of all points \( (x, \alpha) \in S \) is homeomorphic to \( W \).

Now, for each \( x \in I \), \( x \not= 0, 1 \), there exists an integer \( N_x \) such that \( f \) is constant on
\[
U_x = \{ (y, \alpha) : x - 1/N_x < y < x + 1/N_x, \alpha > \alpha_x \}.
\]
If not, then for every integer \( n \), there is a point \( (y_n, \alpha_n) \) such that \( x - 1/n < y_n < x + 1/n \) and \( \alpha_n > \alpha_x \) and \( f((y_n, \alpha_n)) \neq f((x, \alpha_x)) \). But \( x \) is the limit of \( \{ y_n \} \) and some \( \alpha' \in W \) is the limit of \( \{ \alpha_n \} \), so by the continuity of \( f \), \( f((x, \alpha')) \neq f((x, \alpha_x)) \) which is a contradiction since \( \alpha' > \alpha_x \). Similar arguments establish the existence of \( U_0 \) and \( U_1 \).

For each \( U_x \), consider \( U'_x = (x - 1/N_x, x + 1/N_x) \subset I \). The collection \( \{ U'_x : x \in I \} \) is an open cover of \( I \). Pick a finite subcover \( \{ U'_{x_i} \}_{i=1}^k \). Let \( \alpha_{x_1} \) be the largest of the ordinals \( \{ \alpha_{x_i} \}_{i=1}^k \). Then \( f \) is constant on \( B = \bigcup_{\beta > \alpha_{x_1}} S_\beta \).

Extend \( f \) to \( f' : X \rightarrow \{ 0, 1 \} \) by defining \( f'(2) = f(B) \). Clearly \( f' \) is continuous at 2 since \( B \cup \{ 2 \} \) is a neighborhood of 2.

III. Corollary. There exists a zero-dimensional compact space which satisfies Theorem 1.

Proof. Since \( X \) is zero-dimensional, \( \beta X \) is zero-dimensional and \( S \) is dense in \( \beta X \). Since \( F \in C^*(S) \) cannot be extended to \( X \), \( F \) cannot be extended to \( \beta X \). But every two-valued function in \( C^*(S) \) extends to \( X \) and hence to \( \beta X \). So \( \beta X \) is a compact zero-dimensional space which satisfies Theorem 1.

REFERENCES


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