PARTITIONS INTO SPECIFIED PARTS
WHICH APPEAR A SPECIFIED NUMBER OF TIMES

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ABSTRACT. Restrictions of this type, which are known to produce partition identities when any natural number may be used as a part, are shown to produce partition identities when only certain parts may be used.

1. Introduction. Let $N$ be the set of natural numbers and let $A$, $B$, and $S$ be subsets of $N$. $S(n)$ is the number of partitions of $n$ into parts which are elements of $S$. $A^*(n, B)$ is the number of partitions of $n$ into parts which are elements of $B$, such that each part appears exactly $m$ times for some $m \in A$. For example, if $1, 2 \in A$ and $3, 4, 5 \in B$, then $5 + 4 + 3 + 3$ is a partition of 15 of the type enumerated by $A^*(15, B)$.

$A$ is admissible applied to $B$ if there exists an $S$ such that $A^*(n, B) = S(n)$ for every $n \in N$. In this case, $S$ corresponds to $A$. For example, previous results by this author [6] state that $A$ is admissible applied to $N$ if it is one of the following, where $k, d \in N$:

- $A = \{m| m < (2d - 1)k$ and $m \neq (2i - 1)k + j$ with $1 \leq i \leq d - 1$, $0 \leq j \leq k - 1\}$, or
- $A = \{m| m \neq (2i - 1)k + j$ with $1 \leq i \leq d$, $0 \leq j \leq k - 1\}$, or
- $A = \{m| m \neq (2i - 1)k + j$ with $1 \leq i$, $0 \leq j \leq k - 1\}$.

The purpose of this paper is to show that these sets are admissible applied to $B$ for $B \neq N$. In particular, for each set $A$, there is a class of sets such that $A$ is admissible applied to $B$ for every $B$ in this class. In each case, $N$ is a member of the class and, thus, these results generalize the earlier ones.

2. Results.

Theorem 1. Let $k, d \in N$, $B \subseteq N$ such that

$$kB \cap 2kB \subseteq B \cap 2kB \subseteq kB \cup 2dB \subseteq B \cup 2kB.$$  

Then the number of partitions of $n$ into elements of $B$, such that no part appears exactly $(2i - 1)k + j$ times with $1 \leq i \leq d - 1$, $0 \leq j \leq k - 1$ or more than $(2d - 1)k - 1$ times, is equal to the number of partitions of $n$ into elements of $B$. 

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\[ S = [(B \cup 2kB) - (kB \cup 2dkB)] \cup [(B \cap 2kB) - (kB \cap 2dkB)]. \]

A result of Glaisher [4], states that for \( k \in \mathbb{N} \), the set \( A = \{ i \}_{i=1}^{k-1} \) is admissible applied to \( B \). Letting \( d = 1 \) in Theorem 1, we see that this set is admissible applied to \( B \) for every \( B \) such that \( kB \subseteq B \).

**Corollary 1 (Subbarao).** Let \( k \in \mathbb{N} \), \( B \subseteq \mathbb{N} \) such that \( kB \subseteq B \). Then the number of partitions of \( n \) into elements of \( B \), such that parts appear at most \( k - 1 \) times, is equal to the number of partitions of \( n \) into elements of \( S = B - kB \).

Letting \( k = 1 \) in Theorem 1, we obtain a similar result.

**Corollary 2.** Let \( d \in \mathbb{N}, B \subseteq \mathbb{N} \) such that \( dB \subseteq B \). Then the number of partitions of \( n \) into elements of \( B \), such that parts appear exactly \( 2i \) times with \( 1 \leq i \leq d - 1 \), is equal to the number of partitions of \( n \) into elements of \( S = 2[B - dB] \).

When \( B = \{ id \mid j \geq 0, 1 \leq i \leq d - 1 \} \) in Corollary 2, \( A \) is not only admissible applied to \( B \), but the corresponding set is \( A \) itself.

**Corollary 3.** Let \( d \in \mathbb{N}, B = \{ id \mid j > 0, 1 \leq i \leq d - 1 \} \). Then the number of partitions of \( n \) into elements of \( B \), such that each part appears exactly \( 2i \) times with \( 1 \leq i \leq d - 1 \), is equal to the number of partitions of \( n \) into parts of the form \( 2i \) with \( 1 \leq i \leq d - 1 \).

**Theorem 2.** Let \( k, d \in \mathbb{N}, B \subseteq \mathbb{N} \) such that

\[
B \cap 2kB \cap (2d + 1)kB \subseteq kB \cap (4d + 2)kB \subseteq (B \cap 2kB) \cup (B \cap (2d + 1)kB) \cup (2kB \cap (2d + 1)kB) \cup (4d + 2)kB \subseteq B \cap 2kB \cup (2d + 1)kB.
\]

Then the number of partitions of \( n \) into elements of \( B \), such that no part appears exactly \( (2i - 1)k + j \) times with \( 1 \leq i \leq d \), \( 0 \leq j \leq k - 1 \), is equal to the number of partitions of \( n \) into elements of

\[
S = [((B \cup 2kB \cup (2d + 1)kB) - (kB \cup (4d + 2)kB))] \cup [((B \cap 2kB) \cup (B \cap (2d + 1)kB) \cup (2kB \cap (2d + 1)kB)) - (kB \cap (4d + 2)kB)] \cup [B \cap 2kB \cap (2d + 1)kB].
\]

**Corollary 4.** Let \( d, u \in \mathbb{N}, B = \{ 2^i \mid i \geq u - 1 \} \). Then the number of partitions of \( n \) into elements of \( B \), where no part appears exactly \( 2i - 1 \) times with \( 1 \leq i \leq d \), is equal to the number of partitions of \( n \) into elements of \( S = \{ 2^i \mid i \geq u \} \cup \{ (2d + 1)2^{u-1} \} \).
Theorem 3. Let \( k \in \mathbb{N}, B \subseteq \mathbb{N} \) such that \( B \cap 2kB \subseteq kB \subseteq B \cup 2kB \). Then the number of partitions of \( n \) into elements of \( B \), where no part appears exactly \((2i - 1)k + j\) times with \( 1 \leq i, 0 \leq j \leq k - 1 \), is equal to the number of partitions of \( n \) into elements of \( S = [(B \cup 2kB) - kB] \cup [B \cap 2kB] \).

In conclusion, we note that, for all \( k, d, m \in \mathbb{N} \), the set \( B = \lim_{i=1}^{\infty} \) satisfies the requirements of the three theorems of this paper.

3. Proofs.

Proof of Theorem 1. The generating function for \( A^*(n, B) \) is

\[
\Pi_{n \in B} \left[ \frac{1}{1 - x^n} - \sum_{i=1}^{d-1} \sum_{j=0}^{k-1} x^{(2i-1)k+j} - \sum_{j=(2d-1)k}^{\infty} x^{jn} \right]
\]

\[
= \Pi_{n \in B} \left[ \frac{1}{1 - x^n} - \left( \sum_{i=1}^{d-1} x^{(2i-1)k} \right) \left( \sum_{j=0}^{k-1} x^{jn} \right) - \sum_{j=(2d-1)k}^{\infty} x^{jn} \right]
\]

\[
= \Pi_{n \in B} \left[ \frac{1}{1 - x^n} - x^{kn} \left( \frac{1 - x^{(2d-2)kn}}{1 - x^{2kn}} - \frac{x^{(2d-1)kn}}{1 - x^n} \right) \right]
\]

\[
= \Pi_{n \in B} \left( \frac{1}{1 - x^n} \right) \left[ 1 - x^{kn} \left( \frac{1 - x^{(2d-2)kn}}{1 + x^{kn}} \right) \right] \left[ 1 - x^{(2d-1)kn}(1 + x^{kn}) \right]
\]

Now \( kB \cup 2dkB \subseteq B \cup 2kB \). So terms of the forms \( 1 - x^{kn} \) and \( 1 - x^{2dkn} \) with \( n \in B \), cancel with terms of the form \( 1 - x^m \) or \( 1 - x^{2km} \) with \( m \in B \). In fact, \( kB \cap 2dkB \subseteq B \cap 2kB \). So, if for some \( n, n' \in B \), \( 1 - x^{kn} = 1 - x^{2dkn'} \), then both terms cancel with terms \( 1 - x^m = 1 - x^{2km'} \) with \( m, m' \in B \). Also, \( B \cap 2kB \subseteq kB \cup 2kB \). So, if \( 1 - x^m = 1 - x^{2kn'} \) with \( n, n' \in B \), then at least one of these terms cancels with a term of the form \( 1 - x^{km} \) or \( 1 - x^{2dkm} \) with \( m \in B \). Hence, this is the generating function for \( S(n) \) for some \( S \).
The elements of \( N \) which are in \( S \) are those in \((B \cup 2kB) - (kB \cup 2dkB)\) or \((B \cap 2kB) - (kB \cap 2dkB)\).

**Proof of Corollary 1.** When \( d = 1 \), \( 2dkB = 2kB \), and the restriction on \( B \) in Theorem 1 is

\[
kB \cap 2kB \subseteq B \cap 2kB \subseteq kB \cup 2kB \cup kB \cap 2kB,
\]

which is clearly satisfied if \( kB \subseteq B \).

When \( d = 1 \), there is no \( i \) such that \( 1 \leq i \leq d - 1 \), so parts can appear at most \((2d - 1)k - 1 = k - 1 \) times.

Also,

\[
S = [(B \cup 2kB) - (kB \cup 2kB)] \cup [(B \cap 2kB) - (kB \cap 2kB)]
= [(B - kB) - ((B - kB) \cap 2kB)] \cup [(B - kB) \cap 2kB]
= B - kB.
\]

**Proof of Corollary 2.** When \( k = 1 \), \( kB = B \) and \( 2dkB = 2dB \). So, the restriction on \( B \) in Theorem 1 is

\[
B \cap 2dB \subseteq B \cap 2B \subseteq B \cup 2dB \subseteq B \cup 2B,
\]

which is clearly satisfied if \( 2dB \subseteq 2B \); i.e., \( dB \subseteq B \).

When \( k = 1 \), no part can appear \( 2i - 1 \) times with \( 1 \leq i \leq d - 1 \) or more than \( 2d - 2 \) times. So parts can appear \( 2i \) times with \( 1 \leq i \leq d - 1 \).

Also,

\[
S = [(B \cup 2B) - (B \cup 2dB)] \cup [(B \cap 2B) - (B \cap 2dB)]
= [(2B - 2dB) - ((2B - 2dB) \cap B)] \cup [(2B - 2dB) \cap B]
= 2B - 2dB = 2B - dB.
\]

**Proof of Corollary 3.** When \( B = \{id^I | j \geq 0, 1 \leq i \leq d - 1\} \), \( dB = \{id^I | j \geq 1, 1 \leq i \leq d - 1\} \subseteq B \).

Also,

\[
S = 2B - dB = 2B - dB = \{2i | 1 \leq i \leq d - 1\} = A.
\]

**Proof of Theorem 2.** The generating function for \( A^*(n, B) \) is

\[
\prod_{n \in B} \left[ \frac{1}{1 - x^n} - \sum_{i=1}^{d} \sum_{j=0}^{k-1} x^{(2i-1)k+j+i} \right]
= \prod_{n \in B} \left[ \frac{1}{1 - x^n} \left( \sum_{i=1}^{d} x^{2i-1} \right) \left( \sum_{j=0}^{k-1} x^j \right) \right]
\]
\[
\prod_{n \in B} \left[ \frac{1}{1-x^n} - \frac{x^{kn}}{1-x^{2kn}} \right] \left[ \frac{1-x^{2dkn}}{1-x^{2kn}} \right] \left[ \frac{1-x^{2kn}}{1-x^{kn}} \right] = \\
\prod_{n \in B} \left( \frac{1}{1-x^n} \right) \left[ 1-x^{kn} \left( \frac{1-x^{2dkn}}{1+x^{kn}} \right) \right] = \\
\prod_{n \in B} \left( \frac{1}{1-x^n} \right) \left[ \frac{1-x^{kn}}{1+x^{kn}} \right] \left[ 1+x^{(2d+1)kn} \right] = \\
\prod_{n \in B} \left( \frac{1}{1-x^n} \right) \left[ \frac{1-x^{kn}}{1-x^{2kn}} \right] \left[ \frac{1-x^{(4d+2)kn}}{1-x^{(2d+1)kn}} \right].
\]

Now \(kB \cup (4d+2)kB \subseteq B \cup 2kB \cup (2d+1)kB\). So terms of the forms \(1-x^{kn}\) and \(1-x^{(4d+2)kn}\), with \(n \in B\), cancel with terms of the form \(1-x^m\) or \(1-x^{2km}\) or \(1-x^{(2d+1)km}\) with \(m \in B\). In fact,

\(kB \cap (4d+2)kB \subseteq (B \cap 2kB) \cup (B \cap (2d+1)kB) \cup (2kB \cap (2d+1)kB)\).

So, if \(1-x^{kn} = 1-x^{(4d+2)kn}\) with \(n, n' \in B\), then both terms cancel with terms \(1-x^m = 1-x^{2km}\) or \(1-x^{(2d+1)kn} = 1-x^{2km} = 1-x^{(2d+1)km}\) with \(m, m' \in B\).

\((B \cap 2kB) \cup (B \cap (2d+1)kB) \cup (2kB \cap (2d+1)kB) \subseteq kB \cup (4d+2)kB\).

So, if \(1-x^n = 1-x^{2kn}\) or \(1-x^n = 1-x^{(2d+1)kn}\) or \(1-x^{2kn} = 1-x^{(2d+1)kn}\) with \(n, n' \in B\), then at least one of these terms cancels with a term of the form \(1-x^{km}\) or \(1-x^{(4d+2)km}\) with \(m \in B\).

Additionally, \(B \cap 2kB \cap (2d+1)kB \subseteq kB \cap (4d+2)kB\). So, if \(1-x^n = 1-x^{2kn} = 1-x^{(2d+1)kn}\) with \(n, n', n'' \in B\), then two of these terms cancel with terms \(1-x^{km} = 1-x^{(4d+2)km}\) with \(m, m' \in B\). Hence, this is the generating function for \(S(n)\) for some \(S\).

The elements of \(N\) which are in \(S\) are those in

\[(B \cup 2kB \cup (2d+1)kB) - (kB \cup (4d+2)kB)\]
or in

\[((B \cap 2kB) \cup (B \cap (2d+1)kB)) \cup (2kB \cap (2d+1)kB) \subseteq [kB \cup (4d+2)kB]\]
or in \(B \cap 2kB \cap (2d+1)kB\).

Proof of Corollary 4. When \(k = 1\) and \(B = \{2^i \mid i \geq u-1\}\), \(kB = B\), \(2kB = 2B = \{2^i \mid i \geq u\}\), \(2kB = 2d+1)kB = \{(2d+1)2^i \mid i \geq u-1\}\), and \(2kB \cap (2d+1)kB = \{(2d+1)2^i \mid i \geq u\}\). Thus, \(B \cap (2d+1)kB = \emptyset\), \(2kB \cap (2d+1)kB = \emptyset\), and \(2kB \subseteq B\). Hence, the condition
imposed on $B$ in Theorem 2 is
\[ \emptyset \subseteq \emptyset \subseteq 2B \subseteq B \cup (4d + 2)B \subseteq B \cup (2d + 1)B, \]
which is clearly satisfied.

Also,
\[ S = [(B \cup 2kB \cup (2d + 1)kB) - (kB \cup (4d + 2)kB)] \]
\[ \cup [((B \cup 2kB) \cup (B \cup (2d + 1)kB) \]
\[ \cup (2kB \cap (2d + 1)kB)) - (kB \cap (4d + 2)kB)] \]
\[ \cup [B \cap 2kB \cap (2d + 1)kB] \]
\[ = [(B \cup (2d + 1)B) - (B \cup (4d + 2)B)] \cup [2B - \emptyset] \cup [\emptyset] \]
\[ = [(2d + 1)B - (4d + 2)B] \cup 2B \]
\[ = \{(2d + 1)2^i | i = u - 1\} \cup \{2^i | i \geq u\}. \]

Proof of Theorem 3. The generating function for $A^*(n, B)$ is
\[ \prod_{n \in B} \left[ \frac{1}{1 - x^n} - \sum_{i=1}^{\infty} x^{(2i-1)k+j} \right] \]
\[ = \prod_{n \in B} \left[ \frac{1}{1 - x^n} - \left( \sum_{i=1}^{\infty} x^{(2i-1)kn} \right) \left( \sum_{j=0}^{k-1} x^{jn} \right) \right] \]
\[ = \prod_{n \in B} \left[ \frac{1}{1 - x^n} - x^{kn} \left( \frac{1}{1 - x^{2kn}} \right) \left( \frac{1 - x^{kn}}{1 - x^n} \right) \right] \]
\[ = \prod_{n \in B} \left( \frac{1}{1 - x^n} \right) \left( \frac{1 - x^{kn}}{1 + x^{kn}} \right) = \prod_{n \in B} \left( \frac{1}{1 - x^n} \right) \left( \frac{1 - x^{kn}}{1 - x^{2kn}} \right). \]

Now, $kB \subseteq B \cup 2kB$. So, terms of the form $1 - x^{kn}$ with $n \in B$, cancel with terms of the form $1 - x^m$ or $1 - x^{2km}$ with $m \in B$. Also, $B \cap 2kB \subseteq kB$. So, if $1 - x^n = 1 - x^{2kn'}$ with $n, n' \in B$, then one of these terms cancels with a term of the form $1 - x^{km}$ with $m \in B$. Hence, this is the generating function for $S(n)$ for some $S$.

The elements of $N$ which are in $S$ are those in $(B \cup 2kB) - kB$ or in $B \cap 2kB$.

REFERENCES

   MR 46 #8943.

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