ABSTRACT. Johnson [A crinkled arc, Proc. Amer. Math. Soc. 25 (1970), 375—376] has shown that under suitable normalizations all crinkled arcs are unitarily equivalent. Using this result, we find a general series expansion for a crinkled arc:

\[ f(t) = \sqrt{2} \sum_{n=1}^{\infty} x_n \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi}, \]

where \( \{x_n\} \) is an orthonormal set.

Originally introduced in problem four of Halmos [2], a crinkled arc may be defined as a continuous map \( f: [0, 1] \to X \), a Hilbert space, which is one-to-one and possesses the crinkly property: if \( 0 < a < b < c < d < 1 \), then the chords \( f(b) - f(a) \) and \( f(d) - f(c) \) are orthogonal. It is convenient to consider the following normalizations:

(I) \( f(0) = 0 \) by translation,

(II) \( \| f(1) \| = 1 \) by a scale change,

(III) \( X = \sqrt{f} \) where \( \sqrt{f} \) is the smallest Hilbert space containing the values of \( f \).

Under these conditions, Johnson [3] has derived a number of results including \( t \to \| f(t) \| \) is a strictly monotone continuous map of \( [0, 1] \) onto \( [0, 1] \). This allows an additional normalization in the following way: if \( f(t) \) is a crinkled arc with \( \phi(t) = \| f(t) \| \), then \( \hat{f}(t) = f(\phi^{-1}(\sqrt{2}t)) \) represents the same locus but with \( \| \hat{f}(t) \|^2 = t \). Consequently, we introduce

(IV) \( \| f(t) \|^2 = t \)

and consider now only crinkled arcs satisfying (I)–(IV). In this context, Johnson's main result says that any two crinkled arcs are unitarily equivalent in the sense that if \( f: [0, 1] \to X \), \( g: [0, 1] \to Y \), then there is an isometry \( U: X \overset{\text{onto}}{\to} Y \) such that \( g(t) = Uf(t) \). We shall use this result to prove the following representation.

Theorem. \( f(t) \) is a crinkled arc iff

\[ f(t) = \sqrt{2} \sum_{n=1}^{\infty} x_n \frac{\sin(n - \frac{1}{2})\pi t}{(n - \frac{1}{2})\pi}, \]

where \( \{x_n\} \subseteq X \) is an orthonormal set.

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The proof follows from the observation that in the theory of stochastic processes, Brownian motion $W(t)$ defined on $[0, 1]$ may be regarded as a crinkled arc in a suitable Hilbert space $B$ of random variables. A direct application of the Karhunen-Loève expansion theorem provides the representation

$$W(t) = \sqrt{2} \sum_{n=1}^{\infty} b_n \frac{\sin((n - \frac{1}{2})\pi t)}{(n - \frac{1}{2})\pi}$$

where $\{b_n\} \subseteq B$ is an orthonormal set (Ash [1]). If $f(t) \subseteq X$ is a crinkled arc, there is an isometry $U: B \to X$ such that

$$f(t) = UW(t) = \sqrt{2} \sum_{n=1}^{\infty} (Ub_n) \frac{\sin((n - \frac{1}{2})\pi t)}{(n - \frac{1}{2})\pi}.$$

Identifying $x_n = Ub_n$, we have the desired result in one direction. Conversely, if $f(t)$ has a representation (1) then we can define an isometry $U: B \to X$ coordinatewise by $Ub_n = x_n$. It is immediate that all of the properties of $W(t)$ as a crinkled arc are carried into $f(t)$.

**Remark 1.** The series convergence in (1) is uniform in $t$ since

$$\left\| \sqrt{2} \sum_{n=k}^{\infty} x_n \frac{\sin((n - \frac{1}{2})\pi t)}{(n - \frac{1}{2})\pi} \right\|^2 = 2 \sum_{n=k}^{\infty} \|x_n\|^2 \frac{\sin^2((n - \frac{1}{2})\pi t)}{(n - \frac{1}{2})^2\pi^2} \leq 2 \sum_{n=k}^{\infty} \frac{1}{(n - \frac{1}{2})^2\pi^2} \to 0 \text{ as } k \to \infty.$$

**Remark 2.** If (IV) is dropped, then (1) holds with $t$ replaced by $\|f(t)\|^2$ on the right-hand side. Relaxations of (I) and (II) require the obvious modifications.

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**REFERENCES**


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