ON FUNCTIONAL EQUATIONS RELATED TO MIELNIK'S PROBABILITY SPACES

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ABSTRACT. It is shown that the method used by C. V. Stanojevic to obtain a characterization of inner product spaces in terms of a Mielnik probability space of dimension 2 does not admit a generalization to dimension n > 2.

Let \( f: [0, 2] \to [0, 1] \) be continuous and strictly increasing with \( f(0) = 0 \) and \( f(2) = 1 \). The class of all such functions \( f \) will be denoted by \( F \). Likewise, let \( g: [0, 2] \to [0, 2] \) be continuous but strictly decreasing with \( g(0) = 2 \) and \( g(2) = 0 \). Similarly, the class of all such functions \( g \) will be denoted by \( G \). In [1] it is proved that the functional equation

\[
(*) f + f \circ g = 1
\]

where \( (f \circ g)(t) = f(g(t)) \) has a solution \( f \in F \) if and only if \( g \in G \) is an involution, i.e., \( g \circ g = e \) where \( e \) is the identity function on \([0, 2]\). Using this result it is also shown that a normed real linear space \( N \) is an inner product space if and only if for some \( f \in F \), \((S, f(\|x + y\|))\) is a Mielnik probability space [2] of dimension 2. The functional equation \( (*) \) served as a tool to obtain a new characterization of inner product spaces. In this note we consider the possibility of extending this characterization of inner product spaces to the case where \( p \) is a probability function generated by an appropriate function \( f \) and \((S, p)\) is of dimension \( > 2 \).

Let \( g^{(m)} \) denote \( m \) iterations of a function \( g: I \to I \) where \( I \) is some interval. Also, suppose \( g^{(n)} = e \) where \( e \) is the identity function on \( I \) and \( n \) is some positive integer. We shall show that the generalized functional equation

\[
(**) f + f \circ g + f \circ g^{(2)} + \cdots + f \circ g^{(n-1)} = 1
\]

(where \( f \) and \( g \) are functions belonging to a suitable generalization of the classes \( F \) and \( G \) defined earlier) collapses. In other words, the method from [1] cannot be extended in a straightforward manner to the case when \((S, p)\) is of dimension \( > 2 \). The following theorem (for a similar result for homeomorphisms see [3]) is the key to our result:

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Theorem. Let $h: I \rightarrow I$ be a function where $I$ is an interval. If $h$ is continuous and if for some $m \geq 2$, $h^{(m)} = e$, then $h$ is an involution, i.e., $h \circ h = e$.

Proof. Let $h(x) = h(y)$. Then, since $h^{(m)} = e$, we have $h^{(m)}(x) = h^{(m)}(y)$ implies $x = y$ and thus $h$ is one-to-one. Hence, since $h$ is continuous, $h$ is strictly monotone. First we consider the case where $h$ is strictly increasing. Then from $h(x) > x$ it follows that $x = h^{(m)}(x) > h^{(m-1)}(x) > \ldots > h(x) > x$ which is a contradiction. The contradiction also follows from the assumption $h(x) < x$. Hence $h(x) = x$ for all $x$ in $I$ and $h \circ h = e$. Next we consider the case where $h$ is strictly decreasing. If $x < y$, then $h(x) > h(y)$ and $h^{(2)}(x) < h^{(2)}(y)$. Hence $h^{(2)}$ is strictly increasing. But $(h^{(2)})^{(m)} = (h^{(m)})^{(2)} = e^{(2)} = e$. Applying the first case to $h^{(2)}$ we get $h^{(2)} = e$. Therefore $h^{(2)} = e$ and $h$ is an involution.

In particular, our theorem shows that the function $g: I \rightarrow I$ appearing in our generalized functional equation $(**)$ must be an involution. Thus $(**)$ becomes $nf + f \circ g)/2 = 1$ for $n$ even and $(n+1)f/2 + ((n-1)/2)f \circ g = 1$ for $n$ odd. Now if we want to extend the result from [1] to the $n$-dimensional case we have to have $(**)$ since it is equivalent to Axiom (C) of Mielnik [2]. This shows that there is not a trivial extension to dimension $n$ using the procedure from [1].

REFERENCES


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