Abstract. Let $A$ be an $n$-square matrix with zero trace over an algebraically closed field $F$, and let the characteristic of $F$ not divide $n$. It is shown that $A$ can be expressed as $A = XY - YX$ where the eigenvalues of $X$ and $Y$ may be arbitrarily specified as long as those of $X$ are distinct.

Let $F$ be a field, and let $A$ be an $n$-square matrix over $F$. We say that $A$ has property $K$ if the following holds: If $\lambda_1, \lambda_2, \ldots, \lambda_{2n} \in F$ with $\lambda_i \neq \lambda_j$ when $1 \leq i < j \leq n$, then $A$ can be written as a commutator $A = XY - YX$ where $X$ and $Y$ are matrices over $F$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}$, respectively. Clearly if $A$ has property $K$ then $\text{tr}(A) = 0$. C. R. Johnson [2] proved the converse in the case where $F$ is the complex field. We show that every $n$-square nonscalar matrix is similar to a matrix with $n-1$ zero diagonal entries, and then use this result and a theorem due to S. Friedland [1] to extend Johnson's result to arbitrary algebraically closed fields.

Let $A + B$ be the direct sum of the square matrices $A$ and $B$, and let $A(i)$ be the principal submatrix of $A$ that remains after row $i$ and column $i$ are removed. The set of all $n^2$-square matrices over $F$ is denoted by $\Gamma_n(F)$.

**Theorem 1.** If $A \in \Gamma_n(F)$ is not a scalar matrix then there exists $B = (b_{ij}) \in \Gamma_n(F)$ such that $B$ is similar to $A$ and $b_{ii} = 0$ for $i = 1, 2, \ldots, n - 1$.

**Proof.** We induct on $n$. If $A \in \Gamma_n(F)$ is similar to the companion matrix of its characteristic polynomial, then the theorem holds. Hence, the theorem holds for $n = 2$. Suppose that $A \in \Gamma_3(F)$, $A$ is not a scalar matrix, and $A$ is not similar to the companion matrix of its characteristic polynomial. Then $A$ is similar to a matrix of the form

$$
C = \begin{bmatrix}
0 & a & 0 \\
1 & b & 0 \\
0 & 0 & c
\end{bmatrix}
$$
where the polynomial \( \lambda - c \) divides the polynomial \( \lambda^2 - b\lambda - a \). If \( b = c = 0 \), then the theorem holds. If \( b = c \neq 0 \), then \( a = 0 \), and it is easy to see that \( C \) is similar to the matrix

\[
\begin{bmatrix}
0 & -b & b \\
-b & 0 & b \\
-b & -b & 2b
\end{bmatrix}.
\]

Suppose that \( b \neq c \), and let \( D = b + c \). If we apply the theorem to \( D \), we see that there exists a nonsingular \( P \in \Gamma_2(F) \) such that if

\[
B = (b_{ij}) = (1 \pm P)C(1 \pm P)^{-1}
\]

then \( b_{ii} = 0 \) for \( i = 1, 2 \). Hence, the theorem holds for \( n = 3 \). Now suppose that the theorem holds for \( n = m \) where \( m \geq 3 \). Let \( A \in \Gamma_m + 1(F) \) and not be a scalar matrix. Since \( A \) is not a scalar matrix we may assume that \( A(m + 1) \) is not a scalar matrix. Applying the inductive assumption to \( A(m + 1) \), we see that there exists a nonsingular \( P \in \Gamma_m(F) \) such that if

\[
C = (c_{ij}) = (P \pm 1)A(P \pm 1)^{-1}
\]

then \( c_{ii} = 0 \) for \( i = 1, 2, \ldots, m - 1 \). If \( c_{mm} = 0 \) then the theorem follows.

Suppose that \( c_{mm} \neq 0 \). Then \( C(1) \) is not a scalar matrix. Therefore, applying the inductive assumption to \( C(1) \), we see that there exists a nonsingular \( Q \in \Gamma_m(F) \) such that if

\[
B = (b_{ij}) = (1 \pm Q)C(1 \pm Q)^{-1}
\]

then \( b_{ii} = 0 \) for \( i = 1, 2, \ldots, m \). This proves the theorem.

**Theorem 2.** Let \( F \) be an algebraically closed field and let \( \text{char}(F) \nmid n \). If \( A \in \Gamma_n(F) \) with \( \text{tr}(A) = 0 \), then \( A \) has property \( K \).

**Proof.** Clearly the theorem holds for \( A = 0 \). Suppose that \( A \neq 0 \). Since \( \text{tr}(A) = 0 \) and \( \text{char}(F) \nmid n \), \( A \) is not a scalar matrix. Hence, by Theorem 1, since \( \text{tr}(A) = 0 \), there exists a nonsingular \( P \in \Gamma_n(F) \) such that if \( B = PAP^{-1} \) then \( b_{ii} = 0 \) for \( i = 1, 2, \ldots, n \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_{2n} \in F \) with \( \lambda_i \neq \lambda_j \) when \( 1 \leq i < j \leq n \). Let \( U = \lambda_1 \hat{+} \lambda_2 \hat{+} \cdots \hat{+} \lambda_n \), and let \( V = (v_{ij}) \in \Gamma_n(F) \) such that

\[
v_{ij} = b_{ij} / (\lambda_i - \lambda_j), \quad i \neq j, \quad i, j = 1, 2, \ldots, n,
\]

and \( v_{11}, v_{22}, \ldots, v_{nn} \) are chosen [1] so that \( V \) has eigenvalues \( \lambda_{n+1}', \lambda_{n+2}', \ldots, \lambda_{2n}' \). Letting \( X = P^{-1}UP \) and \( Y = P^{-1}VP \), we see that \( A = XY = YX \) where \( X \) and \( Y \) have eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}', \lambda_{n+2}', \ldots, \lambda_{2n}' \), respectively. Therefore, \( A \) has property \( K \).
The requirement that $F$ be algebraically closed cannot be removed unqualifiedly from Theorem 2. To see this, let $A \in \Gamma_2(F)$ such that $A$ has no eigenvalues in $F$. Let $Y \in \Gamma_2(F)$ such that $Y$ has two equal eigenvalues in $F$. Then $Y$ is similar to a matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ where $a, b \in F$. Hence, if $X \in \Gamma_2(F)$, then the matrix $XY - YX$ has two eigenvalues in $F$. Therefore, $A$ does not have property $K$.

The hypothesis that $\text{char}(F) \nmid n$ in Theorem 2 can be replaced by the requirement that $A$ not be a nonzero scalar matrix.

**Theorem 3.** Let $F$ be an algebraically closed field, and let $A \in \Gamma_n(F)$ with $\text{tr}(A) = 0$. Then $A$ has property $K$ if and only if there does not exist a nonzero $a \in F$ such that $A = al$.

**Proof.** If $A$ is not a nonzero scalar matrix, then a slight modification of the proof of Theorem 2 shows that $A$ has property $K$. Suppose that $A = al$ for some nonzero $a \in F$. Assume that $A$ has property $K$. Then $A = XY - YX$ for some $X, Y \in \Gamma_n(F)$ where $X$ has $n$ distinct eigenvalues. Since $A$ is a scalar matrix, we may assume that $X$ is a diagonal matrix. However, if $X$ is a diagonal matrix, then $XY - YX$ has all diagonal entries equal to zero. Since this contradicts $A = al$ where $a \neq 0$, the theorem follows.

Added in proof. Theorem 1 of a paper by Joel Anderson and Joe Parker, Jr. [Linear and Multilinear Algebra 2 (1974), 203–209] which appeared after this note was accepted for publication implies our Theorem 2.

**REFERENCES**


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