MODULI SATISFYING BOTH CHAIN CONDITIONS
WITH RESPECT TO A TORSION THEORY

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ABSTRACT. Goldman [3] has introduced the notion of the length of a module with respect to a torsion theory and has studied finitely-generated modules over left noetherian rings which have finite length. In this note we simplify the proofs of some of Goldman's results and generalize them by removing both the finite-generation and noetherianness conditions.

0. Background and notation. Throughout the following, \( R \) will denote an associative (but not necessarily commutative) ring with unit element \( 1 \). We will denote by \( R\text{-mod} \) the category of all unitary left \( R \)-modules and will abuse notation by writing \( M \in R\text{-mod} \) when we mean to say that \( M \) is an object of \( R\text{-mod} \).

The set of all (hereditary) torsion theories on \( R\text{-mod} \) will be denoted by \( R\text{-tors} \). The reader is referred to [2], [4], [7] for basic information about such theories. If \( r \in R\text{-tors} \) we denote the class of all \( r \)-torsion left \( R \)-modules by \( \mathcal{T}_r \) and the class of all \( r \)-torsion-free left \( R \)-modules by \( \mathcal{T}_r^f \). The \( r \)-torsion subfunctor of the identity functor on \( R\text{-mod} \) is denoted by \( \mathcal{T}_r^f M \). A submodule \( N \) of a left \( R \)-module \( M \) is said to be \( r \)-pure in \( M \) if and only if \( M/N \) is \( r \)-torsion-free. The set \( R\text{-tors} \) is a complete lattice when we define \( r \leq r' \) if and only if \( \mathcal{T}_r \subseteq \mathcal{T}_{r'} \) (or equivalently if and only if \( \mathcal{T}_r^f \supseteq \mathcal{T}_{r'}^f \)) and when, for any subset \( U \) of \( R\text{-tors} \), \( \cap U \) is characterized by \( \mathcal{T}_{\cap U} = \bigcap \mathcal{T}_r \mid r \in U \). For each \( M \in R\text{-mod} \) there is a unique largest element of \( R\text{-tors} \) relative to which \( M \) is torsion-free. We denote this theory by \( \chi(M) \).

Let \( r \in R\text{-tors} \). A nonzero left \( R \)-module \( M \) is said to be \( r \)-cocritical if and only if \( M \) is \( r \)-torsion-free and, for any nonzero submodule \( N \) of \( M \), \( M/N \) is \( r \)-torsion. Such modules are discussed, under various names, in [2] and [8]. Every nonzero submodule of a \( r \)-cocritical left \( R \)-module is \( r \)-cocritical and every \( r \)-cocritical left \( R \)-module is uniform. Moreover, a left \( R \)-module \( M \) is \( r \)-cocritical for some torsion theory \( r \) if and only if it is \( \chi(M) \)-cocritical.

It is not true that every \( r \in R\text{-tors} \) is of the form \( \chi(M) \) for some \( r \)-cocritical left \( R \)-module \( M \). If a torsion theory \( r \) can be so represented then it is said to be prime [2]. Indeed, if \( r \) is prime then \( r = \chi(M) \) for every \( r \)-cocritical left \( R \)-module \( M \). We call the set of all prime torsion theories on \( R\text{-mod} \) the left spectrum of \( R \) and denote it by \( R\text{-sp} \). If \( r \in R\text{-tors} \) we denote by \( \text{pgen}(r) \) the set of all prime torsion theories \( \pi \in R\text{-sp} \) satisfying \( \pi \geq r \). (These are the prime generalizations of \( r \).) It is not always true that, for a given torsion
theory \( \tau \), we have \( \tau = \bigwedge \text{pgen}(\tau) \). When this happens \( \tau \) is said to be semiprime. If \( \tau = \bigwedge U \) for some \( U \subseteq \{ \chi(M) \mid M \in R\text{-mod is } \tau\text{-cricritical} \} \) then \( \tau \) is said to be strongly semiprime. The rings over which every torsion theory is semiprime and those over which every torsion theory is strongly semiprime are characterized in [6]. The latter are precisely the left seminoetherian rings, i.e. those rings having left Gabriel dimension.

Let \( M \) be a left \( R \)-module. We define the support, \( \text{supp}(M) \), of \( M \) to be \( \text{supp}(M) = \{ \tau \in R\text{-sp} \mid M \not\subseteq \bigwedge \tau \} \). See [1] for details. We define the assassin \( \text{ass}(M) \) of \( M \) to be \( \text{ass}(M) = \{ \pi \in R\text{-sp} \mid \text{there exists a } \pi\text{-cricritical submodule of } M \} \). See [1], [2], [8]. For any submodule \( N \) of \( M \), \( \text{ass}(N) \subseteq \text{ass}(M) \subseteq \text{ass}(N) \cup \text{ass}(M/N) \). Moreover, if \( M \) is \( \pi\)-cricritical for some \( \pi \in R\text{-sp} \) then \( \text{ass}(N) = \{ \pi \} \) for any nonzero submodule \( N \) of \( M \).

1. Modules which are both \( \tau \)-artinian and \( \tau \)-noetherian. Let \( \tau \in R\text{-tors} \).

Following Manocha [5], we say that a left \( R \)-module \( M \) is \( \tau \)-artinian [resp. \( \tau \)-noetherian] if and only if the set of all \( \tau \)-pure submodules of \( M \) satisfies the descending chain condition [resp. the ascending chain condition]. The class of all \( \tau \)-artinian [resp. \( \tau \)-noetherian] left \( R \)-modules is a Serre class [5] and hence so is the class of all left \( R \)-modules which are both \( \tau \)-artinian and \( \tau \)-noetherian. We denote this class by \( \mathcal{G}_\tau \). Then \( \bigwedge \tau \subseteq \mathcal{G}_\tau \) and for any left \( R \)-module \( M \), \( M \in \mathcal{G}_\tau \) if and only if \( M/T_\tau(M) \in \mathcal{G}_\tau \).

(1.1) Proposition. If \( \tau, \tau' \in R\text{-tors} \) then:
(1) \( \tau \subseteq \tau' \Rightarrow \mathcal{G}_\tau \subseteq \mathcal{G}_{\tau'} \).
(2) \( \mathcal{G}_\tau \cap \mathcal{G}_{\tau'} = \mathcal{G}_{\tau \wedge \tau'} \).

Proof. (1) Let \( M \in \mathcal{G}_\tau \). If \( M_1 \subseteq M_2 \subseteq \ldots \) is an ascending chain of \( \tau' \)-pure submodules of \( M \) then \( M/M_i \in \mathcal{G}_{\tau'} \subseteq \bigwedge \tau' \) for each index \( i \) and so there exists an index \( k \) for which \( M_k = M_{k+1} = \ldots \). Thus \( M \) is \( \tau' \)-noetherian. Similarly, \( M \) is \( \tau' \)-artinian and so \( M \in \mathcal{G}_{\tau'} \).

(2) By (1), \( \mathcal{G}_{\tau \wedge \tau'} \subseteq \mathcal{G}_\tau \cap \mathcal{G}_{\tau'} \). Conversely, let \( M \in \mathcal{G}_\tau \cap \mathcal{G}_{\tau'} \) and let \( M_1 \subseteq M_2 \subseteq \ldots \) be an ascending chain of \( (\tau \wedge \tau') \)-pure submodules of \( M \). Then \( M/M_j \not\in \bigwedge \tau \cap \bigwedge \tau' \) for each index \( j \) and so we can assume without loss of generality that there are an infinite number of indices \( j \) for which \( M/M_j \) is not \( \tau \)-torsion. By throwing away all of the other links in the chain we can in fact assume that \( M/M_j \) is not \( \tau \)-torsion for all indices \( j \).

For each index \( j \), let \( N_j \) be the submodule of \( M \) defined by \( N_j/M_j = T_\tau(M/M_j) \). Then we have an ascending chain \( N_1 \subseteq N_2 \subseteq \ldots \) of \( \tau \)-pure submodules of \( M \) which must therefore terminate at some index \( k \), i.e. \( N_k = N_{k+1} = \ldots \). For each \( j > k \), we have \( M_j/M_k \subseteq N_j/M_k = N_j/M_k \subseteq \bigwedge \tau \). But \( M_j/M_k \not\subseteq M/M_k \in \bigwedge \tau \cap \bigwedge \tau' \), and so we must have \( M_j/M_k \not\subseteq \bigwedge \tau' \) for all \( j > k \). For each \( j > k \) let \( N'_j \) be the submodule of \( N_j \) defined by \( M_j = T_\tau(M_j/M_k) \). Then we have the ascending chain \( N_k \subseteq N'_{k+1} \subseteq \ldots \) of \( \tau \)-pure submodules of \( N_k \). Since \( M \in \mathcal{G}_\tau \), we
have \( N_k \in \mathcal{Q}_r \) and so there exists an index \( h \) for which \( N_k' = N_{b+1}' = \ldots \). If \( j > h \) then \( M_j/M_h \subseteq N_j'/M_h = N_b'/M_h \in \mathcal{J}_r \), and so \( M_j/M_h \in \mathcal{J}_r \cap \mathcal{J}_r' = \mathcal{J}_r \cap \mathcal{J}_r' \). But \( M_j/M_h \subseteq M/M_h \in \mathcal{F}_r \), and so we must have \( M_j = M_h \). Therefore \( M \) is \((\tau \wedge \tau')\)-noetherian. A similar proof shows that \( M \) is \((\tau \wedge \tau')\)-artinian and so \( M \in \mathcal{Q}_{r \vee \tau'} \). □

(1.2) Corollary. The following conditions are equivalent for \( M \in R\text{-mod} \):

1. \( M \in \mathcal{Q}_r \)
2. \( M \in \mathcal{Q}_r \cap \mathcal{F}_r \) for some \( r \in R\text{-tors} \).

Proof. (1) \( \Rightarrow \) (2): Trivial. (2) \( \Rightarrow \) (1): If \( M \in \mathcal{Q}_r \cap \mathcal{F}_r \), then \( \chi(M) \geq r \) and so \( M \in \mathcal{Q}_X(M) \) by Proposition 1.1(1). □

Goldman [3] has shown that \( M \in \mathcal{Q}_r \) if and only if there exists a chain \( T_r(M) = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \) of submodules of \( M \) having the property that \( M_{i+1}/M_i \) is \( r \)-cocritical for each \( 0 < i < n \). Such a chain is called a \( r \)-chain in \( M \). Moreover, if \( T_r(M) = M_0' \subseteq M_1' \subseteq \ldots \subseteq M_k' = M \) is another \( r \)-chain in \( M \) and then \( k = n \) and

\[ \{ \chi(M_i'/M_i) \mid 0 < i < n \} = \{ \chi(M_j'/M_j) \mid 0 < j < k \}. \]

The integer \( n \) is therefore unique and is called the \( r \)-length of \( M \); it is denoted by \( \text{len}_r(M) \). The set \( \{ \chi(M_i'/M_i) \mid 0 < i < n \} \) of prime torsion theories is called the set of \( r \)-invariants of \( M \) and is denoted by \( \text{inv}_r(M) \). For each \( M \in \mathcal{Q}_r \) we have

\[ \text{len}_r(M) = \text{len}_r(M/T_r(M)) \quad \text{and} \quad \text{inv}_r(M) = \text{inv}_r(M/T_r(M)). \]

Moreover, Goldman [3, Corollary 2.8] showed that if \( M \in \mathcal{Q}_r \) then \( \text{inv}_r(M) = \text{pgen}(r) \cap \text{supp}(M) \). Thus, if \( r \leq r' \) we have \( \text{inv}_r(M) \supseteq \text{inv}_{r'}(M) \).

For each \( M \in R\text{-mod} \) and each \( r \in R\text{-tors} \) let \( W(M, r) \) be the set of all prime torsion theories \( \tau \) satisfying both of the following conditions:

1. \( \tau \) is a minimal element of \( \text{pgen}(r) \);
2. \( \tau \) is a maximal element of \( \text{supp}(M) \).

(1.3) Proposition. If \( r \in R\text{-tors} \) and \( M \in \mathcal{Q}_r \) then \( \text{inv}_r(M) = W(M, r) \).

Proof. We have already noted that \( \text{inv}_r(M) = \text{pgen}(r) \cap \text{supp}(M) \supseteq W(M, r) \). Conversely, if \( \tau \in \text{inv}_r(M) \) and if \( r' \in R\text{-tors} \) satisfies \( r' > \tau \) we want to show that \( r' \notin \text{supp}(M) \). Assume the contrary. Then \( 0 \neq N = M/T_r(M) \in \mathcal{F}_r \subseteq \mathcal{F}_r \subseteq \mathcal{F}_r \). Moreover, \( M \in \mathcal{Q}_r \) implies that \( N \in \mathcal{Q}_r \). If \( 0 = N_0 \subseteq \ldots \subseteq N_k = N \) is a \( r \)-chain in \( N \) then \( N_1 \) is a \( r \)-cocritical submodule of \( N \) and so \( \chi(N_1) \) is a minimal element of \( \text{pgen}(r) \) for if \( \pi'' \in \text{pgen}(r) \) and \( \pi'' \leq \chi(N_1) \), then \( N_1 \in \mathcal{F}_r \chi(N_1) \subseteq \mathcal{F}_r \). On the other hand, for any nonzero submodule \( N' \) of \( N_1 \), \( N_1/N' \in \mathcal{F}_r \subseteq \mathcal{F}_r \). Therefore \( N_1 \) is also \( \pi'' \)-cocritical and so \( \pi'' = \chi(N_1) \). But \( \chi(N_1) \geq \chi(N) \geq r' > \tau \in \text{pgen}(r) \), a contradiction. Therefore we must have \( r' \notin \text{supp}(M) \) and so \( \tau \) is a maximal element of \( \text{supp}(M) \).
We now want to show that \( \pi \) is a minimal element of \( \operatorname{pgen}(r) \). If \( T_r(M) = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \) is a \( r \)-chain in \( M \) then there exists an index \( k, 0 \leq k < n \), for which \( \pi = \chi(M_{k+1}/M_k) \), where \( M_{k+1}/M_k \) is a \( r \)-cocritical left \( R \)-module. But we have already seen that this suffices to prove that \( \pi \) is minimal in \( \operatorname{pgen}(r) \). Therefore \( \pi \in W(M, r) \) and so we have that \( \operatorname{inv}_r(M) = W(M, r) \). \( \square \)

(1.4) Corollary. If \( M \in \mathfrak{A}_r \) for some torsion theory \( r \) then \( M \in \mathcal{F}_r \) if and only if \( W(M, r) = \emptyset \).

(1.5) Proposition. Let \( M \in R \)-mod and let \( r \in R \)-tors be semiprime. Then the following conditions are equivalent:

1. \( M \in \mathfrak{A}_r \).
2. \( W(M, r) \) is a finite set and \( M \in \mathfrak{A}_r \) for each \( \pi \in W(M, r) \).

Proof. (1) \( \Rightarrow \) (2): By Proposition 1.3, (1) implies that \( W(M, r) = \operatorname{inv}_r(M) \) and the latter is a finite set. By Proposition 1.1(1), \( M \in \mathfrak{A}_r \) for each \( \pi \in W(M, r) \). (2) \( \Rightarrow \) (1): Let \( U = \operatorname{pgen}(r)\setminus W(M, r) \). Then \( M \in \mathcal{F}_{AM} \subseteq \mathfrak{A}_U \). Moreover, since \( r \) is semiprime we have \( r = \wedge \operatorname{pgen}(r) = [\wedge U] \wedge [\wedge W(M, r)] \). Since \( W(M, r) \) is a finite set and since \( M \in \mathfrak{A}_r \) for each \( \pi \in W(M, r) \), we have \( M \in \mathfrak{A}_r \) by Proposition 1.1(2). \( \square \)

2. Modules having finite intrinsic length. In Corollary 1.2 we saw the importance of asking when a left \( R \)-module \( M \) belongs to \( \mathfrak{A}_{x(M)} \). Goldman [3] calls modules having this property modules with \textit{finite intrinsic length}.

(2.1) Proposition. If \( 0 \neq M \in \mathfrak{A}_{x(M)} \) then:

1. \( \emptyset \neq \operatorname{ass}(M) = \operatorname{inv}_{x(M)}(M) \);
2. \( \operatorname{ass}(M) \) is a finite set;
3. there exists a large submodule of \( M \) of the form \( \bigoplus_{i=1}^n N_i \), where the \( N_i \) are cyclic cocritical submodules of \( M \);
4. \( M \) has finite uniform dimension;
5. to each \( \pi \in \operatorname{ass}(M) \) we can associate a submodule \( N_{\pi} \) of \( M \) such that the following conditions are satisfied:
   (i) \( \operatorname{ass}(M/N_{\pi}) = \{\pi\} \),
   (ii) \( \operatorname{ass}(N_{\pi}) = \operatorname{ass}(M) \setminus \{\pi\} \);
   (iii) \( 0 = \bigcap N_{\pi} \quad \pi \in \operatorname{ass}(M) \) and this intersection is reduced;
6. \( \chi(M) = \wedge \operatorname{ass}(M) \);
7. \( \chi(M) \) is strongly semiprime.

Proof. We will first prove that for any left \( R \)-module \( M' \in \mathcal{F}_{x(M)} \cap \mathfrak{A}_{x(M)} \) we have \( \operatorname{ass}(M') \subseteq \operatorname{inv}_{x(M)}(M') \). This will prove one inclusion of (1). The proof of the reverse inclusion will be delayed until later. We will proceed by induction on \( \operatorname{len}_{x(M)}(M') \). If \( \operatorname{len}_{x(M)}(M') = 1 \) then \( M' \) is \( \chi(M) \)-cocritical and so \( \operatorname{ass}(M') = \{\chi(M')\} = \operatorname{inv}_{x(M)}(M') \). Now assume that \( \operatorname{len}_{x(M)}(M') = n \)
and that the result has been established for all \( \chi(M) \)-torsion-free modules the \( \chi(M) \)-length of which is less than \( n \). Let \( 0 = M'_0 \subset M'_1 \subset \ldots \subset M'_n = M' \) be a \( \chi(M) \)-chain in \( M' \). Then \( M'/M'_1 \) is \( \chi(M) \)-torsion-free \([3]\) and
\[
\text{len}_{\chi(M)}(M'/M'_1) = n - 1.
\]
Moreover, \( \text{inv}_{\chi(M)}(M') = \{ \chi(M'_1) \} \cup \text{inv}_{\chi(M)}(M'/M'_1) \).

Let \( \pi \in \text{ass}(M') \). There then exists a submodule \( N \) of \( M' \) for which \( \pi = \chi(N) \). If \( N \cap M'_1 \neq 0 \) then \( N \cap M'_1 \) is \( \pi \)-cocritical and so \( \pi = \chi(N \cap M'_1) = \chi(M'_1) \in \text{inv}_{\chi(M)}(M') \). If \( N \cap M'_1 = 0 \) then \( N \cong \{ N + M'_1 \} / M'_1 \subset M'/M'_1 \) and so \( \pi \in \text{ass}(M'/M'_1) \). By induction we then have \( \pi \in \text{inv}_{\chi(M)}(M'/M'_1) \subseteq \text{inv}_{\chi(M)}(M') \).

In particular, we have \( \text{ass}(M) \subseteq \text{inv}_{\chi(M)}(M) \) which is one direction of (1). Since \( \text{inv}_{\chi(M)}(M) \) is a finite set, we immediately have (2).

The module \( M \) certainly has at least one cocritical submodule \( N \) (i.e. \( N \) is \( \tau \)-cocritical for some \( \tau \in R \)-tors), namely the first nonzero link in a \( \chi(M) \)-chain in \( M \). Therefore \( \emptyset \neq \text{ass}(M) \). If \( 0 \neq x \in N \) then \( Rx \) is also cocritical and so without loss of generality we can assume that \( N \) is a cyclic left \( R \)-module. Consider the set of all submodules of \( M \) of the form \( \bigoplus_{i \in \Omega} N_i \) where the \( N_i \) are cyclic cocritical submodules of \( M \). This set is clearly inductive and so by Zorn's lemma it contains a maximal element \( M' = \bigoplus_{i \in \Omega} N_i \).

We claim that \( M' \) is a large submodule of \( M \). Indeed, if not then there exists a nonzero submodule \( M'' \) of \( M \) for which \( M'' \cap M' = 0 \). By Proposition 1.1, \( M'' \in \mathfrak{I}_{\chi(M)} \) and so \( M'' \) has a nonzero cocritical submodule \( N'' \), which we can again assume to be cyclic. Therefore \( M' \oplus N'' \) is strictly larger than \( M' \), contradicting the maximality of \( M' \). Thus \( M' \) is large in \( M \). We now want to show that the set \( \Omega \) is finite. Assume not. Then without loss of generality we can assume that \( \Omega \) contains the set of natural numbers. For each natural number \( i \), let \( Y_i = \bigoplus_{j=1}^i N_j \). Then for each index \( i \), \( M'/Y_i \) is isomorphic to a submodule of \( M' \) and so is \( \chi(M) \)-torsion-free. Hence we have an infinite ascending chain \( Y_1 \subset Y_2 \subset \ldots \) of \( \chi(M) \)-pure submodules of \( M' \). But if \( M \in \mathfrak{I}_{\chi(M)} \), then \( M' \in \mathfrak{I}_{\chi(M)} \), a contradiction. Therefore \( \Omega \) is a finite set and so we have proven (3). As an immediate consequence of this we also have (4).

Now let \( \pi \in \text{ass}(M) \) and consider the class of all submodules \( N' \) of \( M \) for which \( \pi \notin \text{ass}(N') \). This set is clearly nonempty and by \([8\), Proposition 3.1\] it is inductive. Therefore by Zorn's lemma there is a maximal such module \( N_\pi \) which clearly satisfies (5)(ii).

Let \( \pi' \in \text{ass}(M/N_\pi) \). Then there exists a \( \pi' \)-cocritical submodule \( M'/N_\pi \) of \( M/N_\pi \). Thus
\[
\text{ass}(M') \subseteq \text{ass}(N_\pi) \cup \text{ass}(M'/N_\pi) = \text{ass}(N_\pi) \cup \{ \pi' \}
\]
and so, by the maximality of \( N_\pi \), we have \( \pi' = \pi \). This establishes (5)(i).

Now let \( N' \cap \bigcap_{\pi \in \text{ass}(M)} N_\pi \). Then \( \text{ass}(N') \subseteq \text{ass}(N_\pi) \) for all \( \pi \in \text{ass}(M) \) and so \( \text{ass}(N') = \emptyset \). But \( M \in \mathfrak{I}_{\chi(M)} \) implies that \( N' \in \mathfrak{I}_{\chi(M)} \) and
so by what we proved above this implies that $N' = 0$. If $U$ is a proper subset of $\text{ass}(M)$ and if $N'' = \bigcap\{N_\pi | \pi \in U\}$, then $\text{ass}(N'') = \text{ass}(M) \setminus U \neq \emptyset$ and so $N'' \neq 0$. This proves (5)(iii) and so completes the proof of (5).

If $\pi \in \text{ass}(M)$ then $\pi \in \text{inv}_{x(M)}(M) \subseteq \text{pgen}(\chi(M))$. Therefore $\bigwedge \text{ass}(M) \supseteq \chi(M)$. On the other hand, by (5) there exists a canonical monomorphism $M \to M' = \bigoplus\{M/N_\pi | \pi \in \text{ass}(M)\}$. Then $\chi(M) \supseteq \chi(M') = \bigwedge \text{ass}(M)$ by [2, Proposition 5.4] and so $\chi(M) = \bigwedge \text{ass}(M)$, proving (6).

In particular, (6) implies that $\chi(M)$ is semiprime. Since $\text{ass}(M) \subseteq \text{inv}_{x(M)}(M)$, each $\pi \in \text{ass}(M)$ is of the form $\chi(N)$ where $N$ is a $\chi(M)$-cocritical left $R$-module. This implies that $\chi(M)$ is strongly semiprime, proving (7).

Finally, we return to prove the other direction of (1). Let $\pi \in \text{inv}_{x(M)}(M)$. Then $\pi \supseteq \chi(M) = \bigwedge \text{ass}(M)$. By (2), $\text{ass}(M)$ is a finite set of prime torsion theories and so by [3, Lemma 3.5] there exists a $\pi' \in \text{ass}(M)$ for which $\pi \supset \pi'$. But by Proposition 1.3, $\pi$ and $\pi'$ are both maximal elements of $\text{supp}(M)$ and so we must have $\pi = \pi'$. Therefore $\pi \in \text{ass}(M)$ and so $\text{inv}_{x(M)} \subseteq \text{ass}(M)$, proving equality. $\square$

(2.2) **Corollary.** The following conditions are equivalent for $0 \neq M \in R$-mod:

1. $M \in \mathcal{Q}(M)$.
2. (i) $\chi(M)$ is strongly semiprime;
   (ii) $W(M, \chi(M))$ is a finite set;
   (iii) $M \in \mathcal{Q}_\pi$ for each $\pi \in W(M, \chi(M))$.

**Proof.** By Propositions 2.1 and 1.5. $\square$

**REFERENCES**