SOME CHARACTERIZATIONS OF REFLEXIVITY

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ABSTRACT. The results of R. C. James on characterizations of reflexivity of Banach spaces with an unconditional basis in terms of $c_0$ and $l^1$ are extended to arbitrary Banach spaces. Some consequences are obtained.

R. C. James has proved [4, Theorem 2] that a Banach space $E$ with an unconditional basis is reflexive if and only if $E$ contains no subspace isomorphic to $c_0$ or $l^1$. This result has been shown to remain valid for any subspace $E$ of a space with an unconditional basis (of any power) by C. Bessaga and A. Pełczyński [1], [2], who have also proved [2] that such a space $E$ is reflexive if and only if $E^*$ contains no subspace isomorphic to $l^1$.

The above conditions are clearly necessary for the reflexivity of any Banach space $E$. In the present note we shall show that one can add a certain necessary condition to them, which is also satisfied by any (not necessarily reflexive) subspace of a space with an unconditional basis (of any power) in such a way that these conditions together will be also sufficient for reflexivity. Thus, we shall obtain extensions of the above results to characterizations of reflexivity of an arbitrary Banach space $E$.

Following A. Pełczyński [6], a Banach space $E$ is said to have property (u), if for every weak Cauchy sequence $\{x_n\} \subset E$ there exists a sequence $\{y_n\} \subset E$ such that (a) the series $\Sigma_{i=1}^{\infty} y_i$ is weakly unconditionally Cauchy and (b) the sequence $\{x_n - \Sigma_{i=1}^{n} y_i\}$ converges weakly to 0.

Theorem. For a Banach space $E$ the following statements are equivalent:

1. $E$ is reflexive.
2. $E$ has property (u) and $E$ contains no subspace isomorphic to $c_0$ or $l^1$.
3. $E$ has property (u) and $E^*$ contains no subspace isomorphic to $l^1$.

Proof. Assume 1. Then, by [6, Proposition 2], $E$ has property (u). Also, $E^*$ is reflexive and hence $E^*$ contains no subspace isomorphic to $l^1$. Thus, $1 \Rightarrow 3$. 
Assume now $3°$. If $E$ contains a subspace $G$ isomorphic to $c_0$, then $E^*$ has a quotient space $E^*/G$ isomorphic to $l^1$ and hence (see e.g. [3, p. 63, exercise 2]) $E^*$ has a complemented subspace isomorphic to $l^1$, in contradiction with $3°$. On the other hand, if $E$ contains a subspace isomorphic to $l^1$, then so does $E^*$ [7, Proposition 3.3], in contradiction with $3°$. Thus, $3° \Rightarrow 2°$.

Assume now $2°$. Then, since $E$ has property (u) and contains no subspace isomorphic to $c_0$, by [6, Theorem 1] (for a proof see [9, p. 450]), $E$ is weakly complete. Hence, since $E$ contains no subspace isomorphic to $l^1$, from [8, Corollary 1] it follows that $E$ is reflexive. Thus, $2° \Rightarrow 1°$.

Since every Banach space $E$ with an unconditional basis and every subspace of such a space have property (u) (by [6, Theorem 3 and Corollary 1]; for a proof see [9, pp. 445—449]), from the above Theorem we obtain, in particular, the results of R. C. James [4] and C. Bessaga and A. Pełczyński [1], [2] mentioned in the introduction.

**Corollary 1.** A separable Banach space $E$ is reflexive if and only if

(i) $E$ has property (u) and (ii) $E^{**}$ is separable.

**Proof.** Clearly, (i) and (ii) imply $3°$ of the above Theorem.

**Remark 1.** Combining [8] with [6, Corollary 5], it follows that if a Banach space $E$ has property (u) and contains no subspace isomorphic to $l^1$, then $E^*$ is weakly complete. This result yields other proofs of Corollary 1 and the implication $3° \Rightarrow 1°$ of the Theorem.

**Corollary 2.** The following two conjectures are equivalent:

$1°$ [5, p. 165]. Every infinite dimensional Banach space contains an infinite dimensional subspace that is either reflexive or is isomorphic to $c_0$ or $l^1$.

$2°$. Every infinite dimensional Banach space contains an infinite dimensional subspace with property (u),

**Proof.** $c_0$, $l^1$ and every reflexive space have property (u), so $1° \Rightarrow 2°$. Conversely, if $G \subset E$ has property (u), then by the above Theorem either $G$ is reflexive or $G$ contains a subspace isomorphic to $c_0$ or $l^1$. Thus, $2° \Rightarrow 1°$.

**Remark 2.** The conjecture of Corollary 2, if substantiated, would have some interesting consequences, e.g., that every (infinite dimensional) second conjugate space $E^{**}$ contains a reflexive subspace (of infinite dimension)—or, equivalently, that every conjugate Banach space $E^*$ has a reflexive quotient space. Indeed, if $E \supset G$ reflexive, then $E^{**} \supset E \supset G$; if $E \supset c_0$, then $E \supset l^1 \supset l^1$; finally, if $E \supset l^1$, then $E^{**} \supset (l^1)^* \supset l^2$. 

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REFERENCES


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