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ON AN OPERATOR EQUATION INVOLVING MAPPINGS OF MONOTONE TYPE

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ABSTRACT. Let X be a real reflexive Banach space and $A: X \to 2^{X^*}$ a maximal monotone mapping such that the graph $G(A)$ of A is weakly-closed in $X \times X^*$ and $0 \in A(0)$. Also, let $T: X \to 2^{X^*}$ be a quasi-bounded coercive mapping of type (M) such that the effective domain $D(T)$ of T contains a dense linear subspace X_0 of X. Then it is shown that for each $\omega \in X^*$ there exists a $u \in X$ such that $\omega \in Au + Tu$ and the subset $\{u \in X | \omega \in Au + Tu\}$ is a weakly-compact subset of X. An application to an elliptic nonlinear boundary value problem of Neumann type is given.

The aim of this paper is to prove a theorem on the existence of solutions for an equation of type $Au + Tu = f$, where A and T are nonlinear mappings with their domains in a real reflexive Banach space X and range in the dual Banach space X^*. We shall also give an application of this result to the problem of existence of solutions of a nonlinear elliptic boundary value problem of Neumann type in $L^2(\Omega)$, where Ω is a bounded domain in an Euclidean space $\mathbf{R}^n$ with smooth boundary. We may mention that our results also apply to boundary value problems of variational type for quasilinear elliptic systems with strongly nonlinear lower order terms. These applications will appear in a subsequent paper elsewhere. We employ the following definitions: If X is a real reflexive Banach space and X^* is its dual Banach space, we denote by (ω, x) the duality pairing between an element ω in X^* and x in X. For a multivalued mapping $T: X \to 2^{X^*}$ (the set of subsets of X^*), we denote by $D(T)$, *the effective domain of* T, the subset of X defined by $D(T) = \{x \in X | Tx \neq \emptyset\}$. A mapping $T: X \to 2^{X^*}$ is said to be *monotone* if its graph $G(T) = \{[x, \omega] | x \in D(T), \omega \in Tx\}$ is a monotone subset of $X \times X^*$ in the sense that $(\omega_1 - \omega_2, x_1 - x_2) \geq 0$ for all $[x_1, \omega_1] \in G(T)$, $[x_2, \omega_2] \in G(T)$. A monotone mapping $T: X \to 2^{X^*}$ is said to be *maximal monotone* if its graph $G(T)$ is maximal among all the monotone subsets of $X \times X^*$ in the sense of inclusion.

Definition 1. Let X be a reflexive Banach space and X_0 a dense linear subspace of X. Let $T: X \to 2^{X^*}$ be a given mapping with effective domain $D(T)$ such that $X_0 \subset D(T)$. Then T is said to be of *type* (M) *with respect to* X_0 if the following hold:

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(M_1) T is upper semicontinuous from each finite dimensional subspace F of X_0 to X^* endowed with weak topology.

(M_2) For each $x \in D(T)$, Tx is a bounded, closed and convex subset of X^*.

(M_3) Suppose that $\{u_j\}$ is an infinite sequence in X_0, u an element of X and ω an element of X^* such that $\{u_j\}$ converges weakly to u in X (written $u_j \rightharpoonup u$), $\omega_j \in Tu_j$ with $(\omega_j, v) \to (\omega, v)$ for every $v \in X_0$ and $\limsup(\omega_j, u_j) \leq (\omega, u)$. Then $u \in D(T)$ and $\omega \in Tu$.

If in (M_3) we demand, in addition, that $(\omega_j, u_j) \to (\omega, u)$, then T is called a *generalized pseudo-monotone mapping* with respect to X_0. The concept of a generalized pseudo-monotone mapping was originally introduced by Browder [5] and by Browder-Hess [6] who proved results concerning surjectivity of generalized pseudo-monotone mappings (see, e.g., [5, Theorem 7] and [6, Theorem 5]).

Definition 2. Let T be a mapping from X into 2^{X^*} with effective domain $D(T)$. T is said to be *quasi-bounded* if for each $M > 0$ there exists a constant K (depending on M) such that whenever $[x, \omega] \in G(T)$, $(\omega, x) \leq M\|x\|$, $\|x\| \leq M$, then $\|\omega\| \leq K$. Note that a bounded mapping T (which maps bounded subsets of X into bounded subsets of X^*) is clearly a quasi-bounded mapping, while the converse may not be true.

The following proposition shows that for bounded mappings of type (M) we have that $D(T) = X$.

Proposition 1. *Let X be a real reflexive Banach space and X_0 a dense linear subspace of X. Let $T: X \to 2^{X^*}$ be a given mapping with effective domain $D(T)$ such that $X_0 \subset D(T)$. Let T be a bounded mapping of type* (M) *with respect to X_0. Then $D(T) = X$.*

Proof. Let $u \in X$ and $\{u_j\}$ be a sequence in X_0 such that $u_j \rightharpoonup u$ in X. Since T is a bounded mapping, we see that $\bigcup_{j=1}^{\infty} Tu_j$ is a bounded subset of X^* and so, by passing to a subsequence (if necessary), we may take $\omega_j \in Tu_j$ and an $\omega \in X^*$ such that $\omega_j \rightharpoonup \omega$ (weakly) in X^*. Now clearly $\lim_{j\to\infty}(\omega_j, u_j - u) = 0$ so that $\lim_{j\to\infty}(\omega_j, u_j) = (\omega, u)$. It then follows from ($M_3$) that $u \in D(T)$ and $\omega \in Tu$. Hence $X = D(T)$.

This proposition shows that the natural condition on a mapping of type (M) whose effective domain is not all of X is that of quasi-boundedness rather than boundedness. We now state the main result of this paper.

Theorem 1. *Let X be a reflexive (not necessarily separable) Banach space and X_0 a dense linear subspace of X. Let $A: X \to 2^{X^*}$ be a maximal monotone mapping with effective domain $D(A)$, $0 \in A(0)$ and the graph $G(A)$ of A a weakly closed subset of $X \times X^*$. Let $T: X \to 2^{X^*}$ be a quasi-bounded mapping with effective domain $D(T)$, $X_0 \subset D(T)$, of type* (M) *with respect to X_0. Suppose further that the mapping T is coercive, i.e.,*

$\lim_{\omega \in Tu; \|u\| \to \infty}((\omega, u)/\|u\|) = \infty$. *Then the mapping* $A + T$ *from* X *into* 2^{X^*} *is surjective, i.e.,* $R(A + T) = \bigcup_{u \in D(A) \cap D(T)}(Au + Tu) = X^*$. *Moreover, for each* $\omega \in X^*$, *the subset* $\{u \in X \mid \omega \in Au + Tu\}$ *is a weakly compact subset of* X. (*Here* $Au + Tu$ *denotes the vector sum of the subsets* Au *and* Tu *of* X^*.)

Corollary (Brézis [3], Gupta [7], Kenmochi [9]). *Let* X *be a reflexive Banach space,* $A: D(A) \subset X \to 2^{X^*}$ *a linear maximal monotone mapping and* $T: X \to 2^{X^*}$ *a bounded, coercive mapping of type* (M). *Then the range of* $A + T$, $R(A + T) = X^*$, *and for each* $\omega \in X^*$ *the subset* $\{u \in X \mid \omega \in Au + Tu\}$ *is a weakly compact subset of* $X \times X^*$.

Corollary 2 (Browder and Hess [6]). *Let* X *be a reflexive Banach space and* X_0 *a dense linear subspace of* X. *Let* $T: X \to 2^{X^*}$ *be a quasi-bounded mapping, with effective domain* $D(T)$, $X_0 \subset D(T)$, *of type* (M) *with respect to* X_0. *Suppose further that the mapping* T *is coercive, i.e.,* $\lim_{\omega \in Tu; \|u\| \to \infty}((\omega, u)/\|u\|) = \infty$. *Then* $R(T) = X^*$.

Proof of Corollary 2. Take $A: X \to 2^{X^*}$ to be the mapping defined by $Ax = \{0\}$ for each $x \in X$. Then clearly A is a maximal monotone mapping from X into X^* with $0 \in A(0)$, and $G(A) = X \times \{0\}$ a weakly closed subset of $X \times X^*$. The corollary is then an immediate consequence of Theorem 1.

Remark. We may remark that our Theorem 1 is similar to Theorem 1 of Hess [8] where T is a generalized pseudo-monotone mapping and A is a maximal monotone mapping with $[0, 0] \in G(A)$ and 0 is an interior point of $D(A)$, so that A is (strongly) quasi-bounded. Our theorem is, thus, an improvement over Theorem 1 inasmuch as our conditions are more general both on the mapping A and the mapping T. This allows us to consider boundary value problems in L^p-spaces instead of Sobolev-spaces.

Proof of Theorem 1. We first remark, as is standard by now for theorems of this type, that it suffices to prove that $0 \in R(A + T)$. Using a result of Asplund [1], we shall assume in the following that the Banach space X is endowed with a norm in which both X and X^* are strictly-convex. Now, let J be the duality mapping from X^* into X defined for a given $\omega \in X^*$ by $J\omega = u$, where $(\omega, J\omega) = \|\omega\|^2$ and $\|J\omega\| = \|\omega\|$. Such a u exists by Hahn-Banach Theorem and is unique since X is strictly-convex. Also, we have $(\omega_1 - \omega_2, J\omega_1 - J\omega_2) = 0$ for $\omega_1, \omega_2 \in X^*$ which implies that $\omega_1 = \omega_2$ since both X and X^* are assumed to be strictly-convex.

Now, for $\epsilon > 0$ set $A_\epsilon = (A^{-1} + \epsilon J)^{-1}$. It is easy to see that A_ϵ is a single-valued, everywhere defined, bounded maximal monotone demicontinuous (i.e., continuous from X to X^* endowed with weak topology) mapping from X into X^*. Now, let Λ denote the family of finite dimensional subspaces of X_0 and let Λ be partially ordered by inclusion. We may assume that $X_0 = \bigcup\{F \mid F \in \Lambda\}$. For $F \in \Lambda$, let $j_F: F \to X$ denote the inclusion

mapping and $j_F^*: X^* \to F^*$ the corresponding projection mapping. Then the mapping $B_{F\epsilon} = j_F^*(A_\epsilon + T)j_F: F \to 2^{F^*}$ is such that $B_{F\epsilon}(u)$ is a nonempty closed convex subset of F^* for each u in F, and $B_{F\epsilon}$ is upper-semicontinuous from F to F^*. Also, since T is coercive and $(A_\epsilon u, u) \geq 0$ for $u \in X$, we see that there exists an r (independent of ϵ and F) such that $(B_{F\epsilon}u, u) > 0$ for $u \in F$ and $\|u\| \geq r > 0$. It then follows by Proposition 10 of Browder-Hess [6], that for each $\Gamma \in \Lambda$ and $\epsilon > 0$ there exists $u_{F\epsilon} \in F$ such that $\|u_{F\epsilon}\| \leq r$ and $0 \in B_{F\epsilon}u_{F\epsilon}$, or equivalently, there exist $y_{F\epsilon} \in (A_\epsilon + T)u_{F\epsilon}$ such that $y_{F\epsilon} \in F^\perp$ (the annihilator of F in X^*). Now let $\omega_{F\epsilon} \in Tu_{F\epsilon}$ be such that $y_{F\epsilon} = A_\epsilon u_{F\epsilon} + \omega_{F\epsilon}$. Since $(A_\epsilon u_{F\epsilon}, u_{F\epsilon}) \geq 0$ for each $F \in \Lambda$, we get that $(\omega_{F\epsilon}, u_{F\epsilon}) \leq 0$ for $F \in \Lambda$. It then follows from the quasi-boundedness of T that there exists M (independent of F and ϵ) such that $\|\omega_{F\epsilon}\| \leq M$ for $F \in \Lambda$ and $\epsilon > 0$. Since A_ϵ is bounded, and $(y_{F\epsilon}, v) \to 0$ following the filter Λ for each $v \in X_0$, we see that $y_{F\epsilon} \to 0$ (weakly) in X^* following the filter Λ. Now, for each $F \in \Lambda$, let $V_{F\epsilon} = \{[u_{F'\epsilon}, \omega_{F'\epsilon}] \mid F' \in \Lambda, F' \supset F\}$. By our work above, we see that $V_{F\epsilon}$ is contained in some closed ball S in $X \times X^*$ which is weakly compact since X is reflexive. Note that the ball S in $X \times X^*$ does not depend either on F or ϵ. Let $\overline{V}_{F\epsilon}$ denote the weak-closure of $V_{F\epsilon}$ in $X \times X^*$ and let $\epsilon > 0$ be fixed temporarily. The family $\{\overline{V}_{F\epsilon}\}_{F \in \Lambda}$ of weakly closed subsets of the weakly compact set S in $X \times X^*$ clearly has the finite intersection property, and so we get $\bigcap_{F \in \Lambda} \overline{V}_{F\epsilon} \neq \emptyset$. Now, let $[u_\epsilon, g_\epsilon] \in \bigcap_{F \in \Lambda} \overline{V}_{F\epsilon}$. We assert that $u_\epsilon \in D(T)$ so that Tu_ϵ is nonempty and $g_\epsilon \in Tu_\epsilon$. For this, let F_0 be an arbitrary element of Λ. We apply Proposition 11 of Browder-Hess [6] to this $F_0 \in \Lambda$ and the sets $\overline{V}_{F\epsilon}$ to obtain an increasing sequence $F_{j\epsilon}$ in Λ, such that $F_{j\epsilon} \supset F_0$ for each j, and $[u_{j\epsilon}, \omega_{j\epsilon}] \in V_{F_{j\epsilon}}$ such that $u_{j\epsilon} \to u_\epsilon$ (weakly) and $\omega_{j\epsilon} \to g_\epsilon$ (weakly). Let $y_{j\epsilon} = A_\epsilon u_{j\epsilon} + \omega_{j\epsilon}$. Since A_ϵ is bounded and $\{\omega_{j\epsilon}\}, \{u_{j\epsilon}\}$ are bounded, we see that $\{y_{j\epsilon}\}$ is bounded. Also, for each $v \in X_1 = \text{closure} \bigcup_j F_{j\epsilon}$, we have $(y_{j\epsilon}, v) \to 0$ as $j \to \infty$. Since $u_{j\epsilon} \in X_1$ for each j, and a strongly closed subspace of X is also a weakly-closed subspace of X, we see that $u_\epsilon \in X_1$. Now, using the monotonicity of A_ϵ and the fact that $(y_{j\epsilon}, v) \to 0$ for each $v \in X_1$, we have from

$$0 \leq (A_\epsilon u_{j\epsilon} - A_\epsilon u_\epsilon, u_{j\epsilon} - u_\epsilon) = (y_{j\epsilon} - \omega_{j\epsilon} - A_\epsilon u_\epsilon, u_{j\epsilon} - u_\epsilon)$$

that $\limsup(\omega_{j\epsilon}, u_{j\epsilon}) \leq (g_\epsilon, u_\epsilon)$. This gives that $u_\epsilon \in D(T)$ and $g_\epsilon \in Tu_\epsilon$ since T is of type (M) with respect to X_0. Now $[u_\epsilon, g_\epsilon] \in \bigcap_{F \in \Lambda} \overline{V}_{F\epsilon}$ with $g_\epsilon \in Tu_\epsilon$ gives that there is an ultrafilter Λ' of Λ, such that $u_{F\epsilon} \to u_\epsilon$, $\omega_{F\epsilon} \to g_\epsilon$ following Λ'. Also we have that $y_{F\epsilon} \to 0$ following Λ' since Λ' is an ultrafilter of Λ. Now, for $F \in \Lambda'$ we have $u_{F\epsilon} \in A^{-1}(y_{F\epsilon} - \omega_{F\epsilon}) + \epsilon J(y_{F\epsilon} - \omega_{F\epsilon})$. So there exist $v_{F\epsilon} \in A^{-1}(y_{F\epsilon} - \omega_{F\epsilon})$ such that $u_{F\epsilon} = v_{F\epsilon} + \epsilon J(y_{F\epsilon} - \omega_{F\epsilon})$. Using the fact that $0 \in A(0)$ or, equivalently, $0 \in A^{-1}(0)$, we get that

$$(y_{F\epsilon} - \omega_{F\epsilon}, u_{F\epsilon}) \geq \epsilon \|y_{F\epsilon} - \omega_{F\epsilon}\|^2.$$

Since $(y_{F\epsilon}, u_{F\epsilon}) = 0$ for $F \in \Lambda'$ and $\|\omega_{F\epsilon}\| \leq M$, $\|u_{F\epsilon}\| \leq r$, we see that there is a constant C independent of ϵ and F such that $\epsilon \|y_{F\epsilon} - \omega_{F\epsilon}\|^2 \leq C^2$. This then gives that

$$\|u_{F\epsilon} - v_{F\epsilon}\| = \epsilon \|y_{F\epsilon} - \omega_{F\epsilon}\| \leq \sqrt{\epsilon}\, C.$$

So, by passing to another ultrafilter if necessary, we may assume that there exist $h_\epsilon \in X$, $p_\epsilon \in X^*$ such that $y_{F\epsilon} - \omega_{F\epsilon} \to p_\epsilon$, $v_{F\epsilon} \to h_\epsilon$ for $F \in \Lambda'$. Since $G(A^{-1})$ is weakly-closed in $X^* \times X$, we see that $p_\epsilon \in D(A^{-1})$ and $h_\epsilon \in A^{-1}(p_\epsilon)$, or equivalently, $h_\epsilon \in D(A)$ and $p_\epsilon \in Ah_\epsilon$. Since $y_{F\epsilon} \to 0$ in X^* following Λ', and $\omega_{F\epsilon} \to g_\epsilon$ following Λ', we see that $-g_\epsilon = p_\epsilon$. Now using the weak-lower-semicontinuity of the norm in X, we get that $\|u_\epsilon - h_\epsilon\| \leq \sqrt{\epsilon}\, C$ for $\epsilon > 0$. This shows that $\lambda_\epsilon = u_\epsilon - h_\epsilon \to 0$ (strongly) in X as $\epsilon \to 0$. Let $\{\epsilon_n\}$ be a sequence of positive real numbers such that $\epsilon_n \to 0$ as $n \to \infty$. We may assume that there exist $u \in X$ and $g \in X^*$, such that $u_{\epsilon_n} \to u$ and $g_{\epsilon_n} \to g$. Now $h_{\epsilon_n} = u_{\epsilon_n} - \lambda_{\epsilon_n} \to u$ and $Ah_{\epsilon_n} \ni p_{\epsilon_n} = -g_{\epsilon_n} \to -g$ imply that $u \in D(A)$ and $-g \in Au$ since $G(A)$ is weakly-closed in $X \times X^*$. To complete the proof of the theorem, it will now suffice to prove that $g \in Tu$ in order to conclude that $0 \in Au + Tu$. Now, using the monotonicity of A and the fact that $-g \in Au$, we get from $0 \leq (p_{\epsilon_n} + g, h_{\epsilon_n} - u) = (-g_{\epsilon_n} + g, u_{\epsilon_n} - \lambda_{\epsilon_n} - u)$ that $\limsup_{n\to\infty}(g_{\epsilon_n}, u_{\epsilon_n}) \leq (g, u)$, which implies that $u \in D(T)$ and $g \in Tu$ since T is of type (M).

Now to see the weak-compactness of the subset $\{u \in X \mid \omega \in Au + Tu\}$ for a given $\omega \in X^*$, it suffices to note that $\{u \in X \mid \omega \in Au + Tu\}$ is a bounded subset of X since $0 \in A(0)$ and T is coercive. It is then easy to see that the set is weakly-closed under our assumptions if we observe that in a reflexive Banach space the weak-closure of a set consists of limits of weakly-convergent sequences in the set.

This completes the proof of Theorem 1.

Remark. The analogue of Theorem 1 when A is a general maximal monotone mapping still remains unproved. We may note that the condition that $0 \in A(0)$ can be replaced by the condition that $D(A) \cap X_0 \neq \emptyset$, and the coerciveness of T by the coerciveness of $A + T$. We may also remark that, unlike other results of the type of Theorem 1, we do not need to check that $A + T$ is a mapping of type (M). (See, e.g., [5], [6], [8].)

The following theorem is given as an application of Theorem 1 to the problem of existence of solutions of nonlinear boundary value problems of Neumann type in $L^2(\Omega)$.

Theorem 2. *Let Ω be a bounded domain in an Euclidean space $\mathbf{R}^n$ with smooth boundary Γ so that the Sobolev Embedding Theorem holds for Ω. Let β be a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$, such that there exists a convex*

lower-semicontinuous function $j: R \to (-\infty, \infty]$, $j \not\equiv +\infty$ *with* $\beta = \partial j$, *the subdifferential of* j. *Let* $T: L^2(\Omega) \to L^2(\Omega)$ *be a quasi-bounded mapping of type* (M) *such that* $D(T) \supset C_0^\infty(\Omega)$ (*the space of test functions in* Ω) *and* $\lim_{\|u\|_2 \to \infty}((Tu, u)/\|u\|_2) = \infty$. *Then for every* $f \in L^2(\Omega)$ *there exists a* $u \in L^2(\Omega)$ *such that*

$$-\Delta u + Tu = f \quad a.e. \text{ in } \Omega, \qquad -\partial u/\partial n \in \beta(u) \quad a.e. \text{ in } \Gamma.$$

(Here Δ denotes the Laplacian and $\partial/\partial n$ the outward normal.)

Proof. In view of Theorem 1, it suffices to check that the mapping A: $L^2(\Omega) \to L^2(\Omega)$ defined by $Au = -\Delta u$ with effective domain $D(A) = \{u \in H^2(\Omega) | -\partial u/\partial n \in \beta(u) \text{ a.e. in } \Gamma\}$ is a maximal monotone mapping with $0 \in A(0)$ and $G(A)$ weakly-closed in $L^2(\Omega) \times L^2(\Omega)$. That A is maximal monotone follows from Theorem 12 of Brézis [4]. Moreover, we have that there exist constants c_1, c_2 such that $\|u\|_{H^2(\Omega)} \leq c_1 \|-\Delta u + u\|_{L^2(\Omega)} + c_2$ for all $u \in D(A)$. (Here $H^k(\Omega)$ denotes the usual Sobolev space.) Clearly $A(0) = 0$. To check that the graph $G(A) = \{[u, -\Delta u] | u \in D(A)\}$ is weakly closed in $L^2(\Omega) \times L^2(\Omega)$, let $\{u_j\}$ be a sequence in $D(A)$, and $u, g \in L^2(\Omega)$ be such that $u_j \to u$ (weakly) in $L^2(\Omega)$, and $Au_j = -\Delta u_j \to g$ (weakly) in $L^2(\Omega)$. It then follows that the sequence $\{u_j\}$ is bounded in the Sobolev space $H^2(\Omega)$, and so by the Sobolev Imbedding Theorem $u_j \to u$ (strongly) in $L^2(\Omega)$. It is then immediate from the maximal monotonicity of A that $u \in D(A)$ and $g = Au$. Hence, the graph $G(A)$ of A is weakly-closed in $L^2(\Omega) \times L^2(\Omega)$. The theorem then follows immediately from Theorem 1.

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