ON ASYMPTOTIC BEHAVIORS OF ANALYTIC MAPPINGS ON THE MARTIN BOUNDARY

MIKIO NIIMURA

ABSTRACT. Some generalizations of the analogue of Collingwood and Cartwright in the large of Iversen's theorem are given.

Let \( f \) be a nonconstant analytic mapping of a hyperbolic Riemann surface \( R \) into a Riemann surface \( R' \). Let \( R^* \) and \( R'^* \) denote the Martin compactification and any compactification of \( R \) and \( R' \), respectively. \( \Delta \) and \( \Delta' \) denote the Martin ideal boundary of \( R \) and the ideal boundary of \( R' \), respectively. \( \overline{A} \), \( A^c \) and \( \text{int}\ A \) mean the closure, the complement and the interior of a set \( A \) (\( CR^* \) or \( R'^* \)) with respect to \( R^* \) or \( R'^* \), respectively. Let \( \partial A \) denote the relative boundary of \( A \) (\( CR \) or \( R' \)) with respect to \( R \) or \( R' \) and \( f_G \) the restriction of \( f \) to \( G \ (CR) \).

Let \( \{G_n^{(e)}\} \) be a determinant sequence of Kerékkártó-Stoilow's ideal boundary point \( e \), and set \( \Delta_e = \bigcap_n G_n^{(e)} \) and \( \Delta_{G_n^{(e)}} = G_n^{(e)} \cap \Delta \). The cluster set of \( f \) for \( \Delta_e \) is defined by \( C(f, \Delta_e) = \bigcap_n \overline{f(G_n^{(e)})} \), and the range of \( f \) for \( \Delta_e \) by \( R(f, \Delta_e) = \bigcap_n f(G_n^{(e)}) \).

In this paper we assume that the harmonic measure of \( \Delta_e \) is positive.

For \( b \in \Delta_1 \), let \( F_b \) be a filter basis on \( R \) with respect to the fine topology, and set \( \widehat{f}(b) = \bigcup_{U \in F_b} \overline{f(U)} \). Here \( \Delta_1 \) denotes the set of minimal points in \( \Delta \). If \( \widehat{f}(b) \) consists of a single point, then \( \widehat{f}(b) \) is denoted by \( \widehat{f}(b) \).

We say that a curve \( p = (t) (0 < t < 1) \) on \( R \) converges to \( e \), when for every \( n \), there exists \( \lambda(t) \) such that \( \lambda(t) \subset G_n^{(e)} \) for all \( t \geq \lambda(t) \). \( \overline{\lambda(t)} \) denotes the end of this path: \( p = \overline{\lambda(t)} \). Let \( \Gamma(f, \Delta_{G_n^{(e)}}) \) denote the set of asymptotic points along all the paths such that the end of each path is on \( \Delta_{G_n^{(e)}} \), and set \( \chi_f(f, \Delta_e) = \bigcap_n \overline{\Gamma(f, \Delta_{G_n^{(e)}})} \) and \( \chi^*(f, \Delta_e) = \bigcap_n \overline{\Gamma(f, \Delta_{G_n^{(e)}})} \). If for any neighborhood \( V \) of \( \alpha \in R'^* \), \( V \cap \overline{\Gamma(f, \Delta_{G_n^{(e)}})} \) is a nonpolar set, we say \( \alpha \in \Gamma_*(f, \Delta_{G_n^{(e)}}) \) and set \( \chi_*(f, \Delta_e) = \bigcap_n \overline{\Gamma_*(f, \Delta_{G_n^{(e)}})} \).

**Lemma 1.** If \( \alpha \in \chi_*(f, \Delta_e)^c \cap \Gamma(f, \Delta_e) \cap R' \), then \( \alpha \in \text{int} \overline{R(f, \Delta_e)} \).

**Proof.** Since \( \alpha \in \chi_*(f, \Delta_e)^c \cap R' \), there exists a parametric disk \( V \) about \( \alpha \) and \( G_n^{(e)} \) such that \( V \cap \overline{\Gamma(f, \Delta_{G_n^{(e)}})} \) is a polar set. Let \( w = \psi(q) \)
(q ∈ V) be a local parameter of V, and we set \( ψ(V) = \{ w; |w| < 1 \} \), \( ψ(α) = 0 \), \( W_r = \{ w; |w| < r \}, 0 < r < 1 \), \( C_r = ∂W_r \) and \( ψ G_N(e) = g \).

Since \( W_1 \cap \Gamma(g, \Delta_{G_N(e)}) \) is a polar set, its linear measure is zero. Hence \( \Gamma(g, \Delta_{G_N(e)}) \cap C_r = ∅ \) for almost all \( r \) in \( 0 < r < 1 \). Let \( C_r \) have this property and fix \( r \).

Since \( α \in C(f, \Delta_e) \), we see that \( g^{-1}(W_r) \cap G_n^{(e)} \neq ∅ \) for all \( n \geq N \).

If there exists \( G \in \{ G_n^{(e)} \} \) such that \( G \subseteq G_n^{(e)} \) and \( g(G) \subseteq W_r \), then for each point \( b \) of a set \( H_e(\subseteq \Delta_1 \cap \Delta_e) \) whose harmonic measure is positive, \( \hat{h}_G(b) \in W_r \). Indeed the harmonic measure of \( \Delta_e \) is positive and \( g_G \) is a Fatou mapping of \( G \) into \( W_r \).

Hence there exists an asymptotic path \( γ \) from a point of \( G \) to each point \( b \) of \( H_e \) such that \( \lim_{p \to b, p \in \gamma} g_G(p) = \hat{h}_G(b) \). On the other hand, since \( W_r \cap \Gamma(g, \Delta_e) \cap G_n^{(e)}(b) \), for \( b \in H_e \) is a polar set, the harmonic measure of \( H_e \) is zero. This is a contradiction. Thus for all \( n \geq N \), we conclude that \( G_n^{(e)} \cap ∂g^{-1}(W_r) \neq ∅ \).

If \( ∂g^{-1}(W_r) \) contains closed Jordan curves accumulating to \( e \), then we see easily that \( w \in R(g, \Delta_e) \) for any point \( w \) on \( C_r \).

If for all \( n \geq N \), \( G_n^{(e)} \) has at least one noncompact \( γ_n \) of \( g^{-1}(W_r) \), let \( z = φ(p) \) be a local parameter about \( p \in G_n^{(e)} \), and set \( h = g \circ φ^{-1} \). A function element \( Q(w) \) of \( z = h^{-1}(w) \) can be continued analytically along \( C_r \) infinitely often. Indeed if not, when \( w \) tends to a point \( w_1 (\in C_r) \) along \( C_r \), \( γ_n \) is a path whose end is on \( Δ_{G_n(e)} \), and so \( w_1 \in Γ(g, Δ_{G_n(e)}) \). This is a contradiction.

Therefore any point \( w \) on \( C_r \) corresponds to an infinite number of points on \( γ_n \) for any \( n \), and hence \( w \in R(g, \Delta_e) \).

Therefore since for any point \( p \) of \( W_1 \), any neighborhood \( \overline{C W_1} \) of \( p \) contains points of \( R(g, \Delta_e) \), we get \( W_1 \subseteq R(g, \Delta_e) \) and \( α \in R(f, \Delta_e) \), as claimed.

Corollary 1. If \( C(f, \Delta_e) \) is nowhere dense, then \( C(f, \Delta_e) \cap R' \subseteq \chi_*(f, \Delta_e) \).

Proof. If \( α \in \int R(f, \Delta_e) \), for a neighborhood \( V \) of \( α \), any neighborhood \( \overline{C V} \) of any point \( β \in V \) contains at least one point of \( R(f, \Delta_e) \) and \( C(f, \Delta_e) \) is not nowhere dense. Thus we have \( C(f, \Delta_e) \cap R' \subseteq \chi_*(f, \Delta_e) \).

Lemma 2. If \( α \in \chi_*(f, \Delta_e) \cap \chi_+(f, \Delta_e) \cap C(f, \Delta_e) \cap R' \), then \( α \in R(f, \Delta_e) \).

Proof. Suppose that \( α \notin R(f, \Delta_e) \).

Since \( α \in \chi_+(f, \Delta_e) \cap \chi_+(f, \Delta_e) \cap R' \), there exist a parametric disk \( U \) about \( α \) and \( G_n^{(e)} \) such that \( U \cap \Gamma(f, Δ_{G_n^{(e)}}) \) is a polar set and \( α \notin \Gamma(f, Δ_{G_n^{(e)}}) \).

All the \( α \)-points of \( f_{G_n^{(e)}} \) are contained in a finite set of parametric
disks \{U_k\} (k = 1, 2, \ldots, L) such that \( U_i \cap U_j = \emptyset \) \((i \neq j)\). Let \( V \) be a parametric disk about \( \alpha \) satisfying \( V \subset (\bigcap_{k=1}^{L} f_{G_n}^{-1}(U_k)) \cup U \). We fix \( r \) such that \( \Gamma(g, \Delta_{G_n}(e)) \cap C_r = \emptyset \). There exists a diameter \( d_r \) of \( W_r \) such that \( \Gamma(f, \Delta_{G_n}(e)) \cap d_r = \emptyset \). There exists a diameter \( d_r \) of \( W_r \) such that \( \Gamma(f, \Delta_{G_n}(e)) \cap d_r = \emptyset \).

Since \( b \in R(f, \Delta_e) \) for \( b \in C_r \), there exists a connected component \( D \) of \( g^{-1}(W_r) \) which is not relatively compact. Choose a point \( p \) on \( \partial D \) which is mapped by \( g \) to an endpoint of \( d_r \). The function element \( Q(w) \) corresponding to \( p \) can be continued analytically along \( d_r \) through the point 0 to the antipodal point and \( d_r \) is mapped on a cross-cut of \( D \). But on the other hand, since \( D \) does not contain the zeros of \( g \), we have a contradiction, and we conclude that \( \alpha \in R(f, \Delta_e) \).

**Theorem 1.** If \( R^* \) is a metrizable and resolutive compactification of \( R' \) and, for at least one \( n \), \( \Gamma(f, \Delta_{G_n}(e)) \) is a polar set, then \( R(f, \Delta_e)^c \cap R' \subset \chi(f, \Delta_e) \).

**Proof.** From Lemma 2, we have \( R(f, \Delta_e)^c \cap R' \subset \chi(f, \Delta_e) \cup \chi(f, \Delta_e) \)

If \( C(f, \Delta_e)^c \neq \emptyset \), there exist a parametric disk \( V \) and \( G \in \{G(e)\} \) (\( G \subset G_n \)) such that \( f(G) \cap \overline{V} = \emptyset \). Since the mapping \( f_G \) of \( G \) into \( R' - \overline{V} \) is a Fatou mapping, it contradicts that the harmonic measure of \( H_e \) is positive, as we see from the proof of Lemma 1.

Thus from \( \Gamma(f, \Delta_{G_n}(e)) = \emptyset \), we have \( R(f, \Delta_e)^c \cap R' \subset \chi(f, \Delta_e) \).

**Lemma 3.** If \( \alpha \in \chi^*(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R' \), then \( \alpha \in \text{int} R(f, \Delta_e) \).

**Proof.** In Lemma 1, take "all \( r \) in \( 0 < r < 1 \)" instead of "almost all \( r \) in \( 0 < r < 1 \)" and consider "\( W_1 \cap \Gamma(g, \Delta_{G_n}(e)) = \emptyset \)" instead of "\( W_1 \cap \Gamma(g, \Delta_{G_n}(e)) \) is a polar set"; then we have \( w \in R(g, \Delta_e) \) for all \( w \in R(g, \Delta_e) \) as in the proof of Lemma 1.

If \( w_0 \in W_{r/2} \) \((w_0 \neq 0)\), we have \( w_0 \in C(g, \Delta_e) \) and \( W_{r/2} \cap \Gamma(g, \Delta_{G_n}(e)) = \emptyset \) \((W_{r/2} = \{w; |w - w_0| < r/2\})\), and hence \( 0 \in R(g, \Delta_e) \).

Thus we have \( W_1 \subset R(g, \Delta_e) \) and \( \alpha \in \text{int} R(f, \Delta_e) \).

**Theorem 2.** \( R(f, \Delta_e)^c \cap C(f, \Delta_e) \cap R' \subset \chi^*(f, \Delta_e) \).

**Proof.** From Lemma 3, we have

\[ \chi^*(f, \Delta_e)^c \subset C(f, \Delta_e)^c \cup R'^c \cup (\text{int } R(f, \Delta_e)) ; \]

that is,
Lemma 4. \( \text{int} C(\alpha, \Delta_e^c) \subseteq R(\alpha, \Delta_e^c). \)

Proof. If \( \alpha \in \text{int} C(\alpha, \Delta_e^c), \) for any neighborhood \( U \) of \( \alpha, \) there exists a parametric disk \( V_0 \) about \( \alpha_0 \) satisfying \( \overline{V_0} \subseteq U \cap C(\alpha, \Delta_e^c). \) Since \( \alpha_0 \in C(\alpha, \Delta_e^c), \) there exists \( p_1 \in G_1^{(e)} \) such that \( \alpha_1 = f(p_1) \in V_0. \) We can take a parametric disk \( V_1 \) about \( \alpha_1 \) satisfying \( \overline{V_1} \subseteq V_0 \cap f(G_1^{(e)}). \) Repeating the same method, we have a sequence of parametric disks \( \{V_n\} (n = 1, 2, 3, \ldots) \) such that \( \overline{V_{n+1}} \subseteq V_n \) and \( \overline{V_n} \subseteq f(G_n^{(e)}). \beta \in \bigcap_n \overline{V_n} \) is assumed by \( f \) in any \( G_{n}^{(e)}, \) and hence \( \alpha \in R(\alpha, \Delta_e^c). \)

Corollary 2. \( R(\alpha, \Delta_e^c)^c \cap R' \subseteq \chi^*(\alpha, \Delta_e^c) \) if and only if \( \overline{R(\alpha, \Delta_e^c)} = R'^* \).

Proof. If \( C(\alpha, \Delta_e^c) \neq R'^*, \) there exists \( \alpha_0 \) such that \( \alpha_0 \in C(\alpha, \Delta_e^c)^c \cap R' \subseteq \overline{R(\alpha, \Delta_e^c)^c} \cap R'. \) If \( \alpha \in \chi^*(\alpha, \Delta_e^c), \) then we have \( \alpha \in \overline{R(\alpha, \Delta_e^c)} \) for any \( n \) and \( 0 \in \Gamma(g_n^{(e)}) \) for a parametric disk \( V \) about \( \alpha. \) Since there exists \( w_n \in W_{1/n}^* \cap \Gamma(g_n^{(e)}), \) there exists \( p_n \in G_n^{(e)} \) such that \( g(p_n) \in W_{1/n}. \) Since \( p_n \) converges to \( e \) and \( g(p_n) \) converges to \( 0, \) we see that \( 0 \in C(\alpha, \Delta_e^c) \) and \( \alpha \in C(\alpha, \Delta_e^c). \) Hence we have \( \chi^*(\alpha, \Delta_e^c) \subseteq C(\alpha, \Delta_e^c) \) and \( \alpha_0 \notin \chi^*(\alpha, \Delta_e^c). \) Thus if \( R(\alpha, \Delta_e^c)^c \cap R' \subseteq \chi^*(\alpha, \Delta_e^c), \) from Lemma 4, \( \overline{R(\alpha, \Delta_e^c)} = R'^*. \)

Conversely if \( \overline{R(\alpha, \Delta_e^c)} = R'^*, \) then we have, from Theorem 2,

\[ R(\alpha, \Delta_e^c)^c \cap R' = R(\alpha, \Delta_e^c)^c \cap C(\alpha, \Delta_e^c) \cap R' \subseteq \chi^*(\alpha, \Delta_e^c). \]

Corollary 3. If the characteristic function of \( f \) (cf. [3]) is unbounded, then \( R(\alpha, \Delta_e^c)^c \cap R' \subseteq \chi^*(\alpha, \Delta_e^c). \)

Proof. If \( C(\alpha, \Delta_e^c) \neq R'^*, \) since \( f_G \) is a Lindel"of mapping, as in the proof of Theorem 1, the characteristic function of \( f \) is bounded, and a contradiction. Thus from Lemma 4 and Corollary 2 we get \( R(\alpha, \Delta_e^c)^c \cap R' \subseteq \chi^*(\alpha, \Delta_e^c). \)

REFERENCES


DEPARTMENT OF MATHEMATICS, SHIBAURA INSTITUTE OF TECHNOLOGY, 3-9-14, SHIBAURA, MINATO-KU, TOKYO, JAPAN