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CHARACTERIZING hol(Ω)

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ABSTRACT. We find necessary and sufficient conditions for an algebra of complex valued functions defined on a σ-compact T_2 space to be algebraically and topologically equivalent to the algebra of analytic functions on a finitely connected domain in C.

Let Ω be a finitely connected open set in the complex plane $\mathbf{C}$. Denote by hol(Ω) the set of all functions which are analytic on Ω. Note that hol(Ω) is an algebra with respect to the operations of pointwise addition and multiplication of functions and scalar multiplication by complex numbers. Note also that hol(Ω) is a Fréchet space with respect to the topology of uniform convergence on compact subsets of Ω.

Let X be a σ-compact T_2 space. That is, $X = \bigcup_1^\infty X_i$ where X_i is a compact subset of X. Let A be an algebra of complex valued continuous functions on X which is complete with respect to the topology of uniform convergence on the sets X_i and contains the constant functions. We give conditions on A which are necessary and sufficient for A to be topologically and algebraically isomorphic to hol(Ω) for some finitely connected open set Ω in $\mathbf{C}$.

In order to find conditions on A which are natural, we will first consider hol(Ω) and see what kind of conditions this algebra admits. If $C_1, \ldots, C_n$ are the bounded components of $\mathbf{C} - \Omega$ and z_i is in C_i, then polynomials in the functions $f, f_1, \ldots, f_n$, $f(z) = z$ and $f_i(z) = (z - z_i)^{-1}$ are dense in hol(Ω).

A derivation on an algebra B is a linear function $d: B \longrightarrow B$ which satisfies the multiplicative condition $d(ab) = d(a)b + ad(b)$ for every a and b in B. There is a natural derivation on hol(Ω) given by $d(g) = g'$ (g' denotes the ordinary derivative of g).

We will show that the conditions outlined in the above two paragraphs are necessary and sufficient for the algebra A to be topologically and algebraically isomorphic to hol(Ω). This result is given in the following

Theorem. *The following two conditions are necessary and sufficient for A to be topologically and algebraically isomorphic to* hol(Ω) *for some finitely connected open set Ω in C.*

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1. *There is a function f in A and complex numbers $z_1, \ldots, z_n$ in the resolvent set of f such that polynomials in f, $(f - z_1)^{-1}, \ldots, (f - z_n)^{-1}$, are dense in A.*

2. *There is a derivation $d: A \to A$ such that $d(f)$ is never zero on X.*

Proof. We noted in the paragraphs preceding the Theorem that conditions 1 and 2 are necessary. We will prove they are also sufficient.

Without loss of generality we can assume $X = \bigcup_1^\infty X_i$ where X_i is compact and $X_i \subset X_{i+1}$. We let M denote the collection of all continuous homomorphisms from A onto $\mathbf{C}$. Note that $M = \bigcup_1^\infty M_i$ and $M_i \subset M_{i+1}$ where M_i is the set of all homomorphisms m in M which satisfy the continuity condition $|m(g)| \le \sup_{x \in X_i} |g(x)|$ for every g in A. For each g in A define a function $\hat{g}: M \to \mathbf{C}$ by $\hat{g}(m) = m(g)$ ($\hat{g}$ is the Gelfand transform of g). Since for each x in X the map $g \to g(x)$ is a homomorphism of A onto $\mathbf{C}$, we have that A is topologically and algebraically isomorphic to the algebra $\hat{A}$ consisting of all functions of the form $\hat{g}$, g in A and having the topology of uniform convergence on the sets $\{M_i\}$.

Now consider the set $\hat{f}(M)$. It follows from the continuity of the elements of M and the fact that A is generated by $f, f_1, \ldots, f_n$ that $\hat{f}$ is one-to-one. We define a set Ω in $\mathbf{C}$ by $\Omega = \hat{f}(M)$. For each g in A define a function $\dot{g}$ on Ω by $\dot{g}(z) = \hat{g}(\hat{f}^{-1}(z))$. Let $\dot{A}$ denote the algebra of all functions $\dot{g}$, g in A. The map $g \to \dot{g}$ is a topological and algebraic isomorphism of A onto $\dot{A}$ when $\dot{A}$ is given the topology of uniform convergence on the sets $\{\hat{f}(M_i)\}$.

The algebra A is generated by the functions $f, f_1, \ldots, f_n$. For the function f we have $\dot{f}(z) = \hat{f}(\hat{f}^{-1}(z)) = z$. Hence $\dot{A}$ is the completion of the polynomials in z, $(z - z_i)^{-1}, \ldots, (z - z_n)^{-1}$ on Ω with respect to the topology of uniform convergence on the sets $\{\hat{f}(M_i)\}$. If we can show that $\Omega = \bigcup \operatorname{int} \hat{f}(M_i)$, then we will have that Ω is open and that the topology of uniform convergence on the sets $\{\hat{f}(M_i)\}$ is the same as the topology of uniform convergence on compact subsets of Ω. It will then follow that $\dot{A} \subset \operatorname{hol}(\Omega)$. If we can show that each bounded component of $\mathbf{C} - \Omega$ contains one of the points z_i, $i = 1, \ldots, n$, then we can conclude that $\dot{A} \supset \operatorname{hol}(\Omega)$.

We will first show that $\Omega = \bigcup \operatorname{int} \hat{f}(M_i)$. Assume the contrary. That is, there is a z_0 in $\Omega - \bigcup \operatorname{int} \hat{f}(M_i)$. Using the identification we have made between A and $\dot{A}$, we can regard d as a derivation on $\dot{A}$. We proved in a previous paper [2] that such a derivation is necessarily continuous. Let $\|\dot{g}\|_n = \max |\dot{g}(\hat{f}(M_n))|$. Since d is linear and continuous, there is a natural number p and a constant K such that $|d(\dot{g})(z_0)| \le K\|\dot{g}\|_p$ for every $\dot{g}$ in $\dot{A}$. The set $\hat{f}(M_p)$ is the spectrum of f in the algebra which is the completion of A with respect to the sup-norm on X_p. Hence $\hat{f}(M_p)$ is a compact subset of $\mathbf{C}$ with at most $n+1$ components in its complement and each of the bounded components contains one of the points z_i, $i = 1, \ldots, n$. Let b denote the point derivation defined on $\dot{A}$ by $\mathrm{b}(g) = d(\dot{g})(z_0)$. Since $|d(\dot{g})(z_0)| < K\|\dot{g}\|_p$ for all $\dot{g}$ in $\dot{A}$, we can extend b to a continuous derivation on $\dot{A}_p$, the completion of $\dot{A}$, with respect to the seminorm $\|\cdot\|_p$.

Since z_0 is a boundary point of $\hat{f}(M_p)$, there is a function g_0 in $C(\hat{f}(M_p))$ such that g_0 is analytic on int $\hat{f}(M_p)$ and $|g_0(z_0)| > |g_0(z)|$ for all $z \neq z_0$ in $\hat{f}(M_p)$. It follows from the properties of rational approximation (see [3]) that g_0 can be uniformly approximated on $\hat{f}(M_p)$ by polynomials in $f, f_1, \ldots, f_n$. Therefore g_0 is in $\dot{A}_p$. Since g_0 peaks at the point z_0, we have that b must be zero on all of $\dot{A}_p$ (see [1, Corollary 1.6.7]). Since $\hat{f}(M_p)$ is the spectrum of f with respect to the algebra A_p and z_0 is in the boundary of $\hat{f}(M_p)$, there is a x_0 in X such that $z_0 = \hat{f}(x_0)$. We have shown that $0 = \mathrm{b}(\dot{f}) = d(f)(x_0)$, contradicting our assumption that $d(f)$ is never zero on X. Therefore we must have $\Omega = \bigcup \operatorname{int} \hat{f}(M_i)$ and $A \subset \mathrm{hol}(\Omega)$.

All that remains is to show that each bounded component of $\mathbf{C} - \Omega$ contains one of the points z_i, $i = 1, 2, \ldots, n$. Suppose there is a bounded component C which does not contain any of the z_i. The boundary of C is a compact set. Since $\Omega = \bigcup \operatorname{int} \hat{f}(M_i)$, we have that $C \subset \hat{f}(M_j)$ for some j. The functions in $\dot{A}$ have unique analytic extensions to $\Omega \cup C$ and $\max|\dot{g}(\hat{f}(M_j) \cup C))| = \max|\dot{g}(\hat{f}(M_j))|$ for every $\dot{g}$ in $\dot{A}$. Let y be a point in C. The homomorphism $m\colon A \to \mathbf{C}$, defined by $m(g) = \dot{g}(y)$ for g in A, is continuous. This contradicts the fact that $\Omega = \hat{f}(M)$. Hence, every component of $\mathbf{C} - \Omega$ contains one of the z_i, and we have $\dot{A} = \mathrm{hol}(\Omega)$.

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