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A TOTALLY REAL SURFACE IN CP^2 THAT IS NOT TOTALLY GEODESIC

GERALD D. LUDDEN,[1] MASAFUMI OKUMURA AND KENTARO YANO

ABSTRACT. An example of a totally real surface immersed in complex projective space is given. This surface is not totally geodesic. The relation of this example to previous theorems on totally real submanifolds is given.

1. **Introduction.** Let CP^m denote m-dimensional complex projective space, normalized so that CP^m is of constant holomorphic sectional curvature 4. Let J denote the almost complex structure of CP^m. A submanifold M immersed in CP^m is said to be *totally real* if $M_x \cap JM_x = \{0\}$ for all x in M, where M_x denotes the tangent space to M at x. There have recently been several papers concerned with totally real submanifolds of complex manifolds (see [1], [2], [4], [8]). In particular, Chen and Ogiue [2] have proved

Theorem 1. *Let M be a compact n-dimensional manifold isometrically immersed in CP^n as a minimal, totally real submanifold. If $\|\sigma\|^2 < (n+1)/(2-1/n)$, then M is totally geodesic. Here σ is the second fundamental form of the immersion.*

The purpose of this paper is to show that there exist submanifolds of CP^n for which the above inequality becomes equality and to determine such submanifolds. The existence of such submanifolds relies on the examples of minimal submanifolds of spheres given in Chern-do Carmo-Kobayashi [3].

2. **An example of a totally real submanifold.** Let $S^p(r)$ denote the Euclidean sphere of dimension p and radius r. We denote $S^p(1)$ by just S^p. It is now well known that S^{2m+1} is a principal circle bundle over CP^m [7]. This is the Hopf fibration. If we consider S^{2m+1} as a hypersurface of the Euclidean space E^{2m+2} equipped with its natural Kaehler structure $\tilde{J}$, then $\xi = \tilde{J}N$, where N is the outward unit normal to S^{2m+1}, is a unit tangent vector field on S^{2m+1} whose integral curves are great circles. These great circles are the fibres of the Hopf fibration. Let $\pi: S^{2m+1} \to CP^m$ denote the submersion of this fibration.

Let R_1, R_2, R_3 denote 3 copies of the Euclidean plane E^2 and S_1, S_2, S_3 denote the sphere $S^1(1/\sqrt{3})$ in each of these planes, respectively. Then,

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as mentioned in [3], $S_1 \times S_2 \times S_3$ is naturally immersed in S^5 as a minimal submanifold. Of course, $S_1 \times S_2 \times S_3$ is a flat manifold and if $\bar{\sigma}$ is the second fundamental form of the immersion, it can be checked that (see [3]) $\|\bar{\sigma}\|^2 = 6$. If N_a denotes the unit outward normal on S_a $(a = 1, 2, 3)$, then $N = (N_1, N_2, N_3)$ is the outward normal to S^5 at a point of $S_1 \times S_2 \times S_3$ and it is easy to see that $\xi = \tilde{J}N$ is tangent to $S_1 \times S_2 \times S_3$ at this point. Here $\tilde{J}$ is the natural almost complex structure on E^6. Since each of the factors S_1, S_2, S_3 can be assumed to lie in complex subspaces of E^6, just as in [5], there is a submanifold T^2 of CP^2 and a submersion $\pi : S_1 \times S_2 \times S_3 \to T^2$ such that the following diagram is commutative

$$\begin{tikzcd} S_1 \times S_2 \times S_3 \arrow[r, "\bar{\iota}"] \arrow[d, "\pi"'] & S^5 \arrow[d, "\tilde{\pi}"] \\ T^2 \arrow[r, "\iota"] & CP^2 \end{tikzcd}$$

It is also shown in [5] that ι is a minimal immersion.

Let (ϕ, η, ξ) denote the almost contact structure on S^5 induced from $\tilde{J}$. Then

$$\begin{aligned} \phi(\tilde{J}(N_1, 0, 0)) &= \tilde{J}^2(N_1, 0, 0) - \eta(\tilde{J}(N_1, 0, 0))N \\ &= -(N_1, 0, 0) - \eta(\tilde{J}(N_1, 0, 0))N, \end{aligned}$$

which is normal to $S_1 \times S_2 \times S_3$. Similarly $\phi(\tilde{J}(0, N_2, 0))$ and $\phi(\tilde{J}(0, 0, N_3))$ are normal to $S_1 \times S_2 \times S_3$. However, $\{\tilde{J}(N_1, 0, 0), \tilde{J}(0, N_1, 0), \tilde{J}(0, 0, N_3)\}$ forms a basis of tangent vectors to $S_1 \times S_2 \times S_3$. We say $S_1 \times S_2 \times S_3$ is an *anti-invariant* (under ϕ) submanifold of S^5. It can now be easily checked that T^2 is a totally real submanifold of CP^2 (see [6] for the local basis of all four manifolds that shows this). Since T^2 is totally real, formula (4.6) of [7] becomes $\|\bar{\sigma}\|^2 = \|\sigma\|^2 + 2 \cdot 2$, where σ is the second fundamental form of the immersion ι. Thus, T^2 is a compact, minimal, totally real surface immersed in CP^2 with $\|\sigma\|^2 = 2$. But this is precisely the case of equality in the theorem of Chen and Ogiue for $n = 2$.

Remark. Let Z_a be the complex coordinates of R_a $(a = 1, 2, 3)$. Then $S^5 = \{(Z_1, Z_2, Z_3) \mid |Z_1|^2 = |Z_2|^2 + |Z_3|^2 = 1\}$ and $S_1 \times S_2 \times S_3 = \{(Z_1, Z_2, Z_3) \mid |Z_1|^2 = |Z_2|^2 = |Z_3|^2 = 1/3\}$. Basically $\tilde{\pi}$ and π are defined by the identification $(Z_1, Z_2, Z_3) \sim (Z_1/Z_3, Z_2/Z_3)$. Then $|Z_1/Z_3|^2 = |Z_2/Z_3|^2 = 1$ for (Z_1, Z_2, Z_3) in $S_1 \times S_2 \times S_3$ so $T^2 = S^1 \times S^1$.

3. **Main theorems.** Let M be an n-dimensional totally real, minimal submanifold of CP^n. We choose a local field of orthonormal frames $e_1, \ldots, e_{2n}$ in CP^n such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and $\omega^1, \ldots, \omega^{2n}$ is the field of dual frames. Then we have $d\omega^A = -\Sigma\omega^A_B \wedge \omega^B$, $\omega^A_B + \omega^B_A = 0$ $(A, B = 1, -, 2n)$, from the structure equations of CP^n. Restrict-

ed to M, we have $\omega^{n+1} = \cdots = \omega^{2n} = 0$. Thus, $0 = d\omega^\alpha = -\Sigma\,\omega_i^\alpha \wedge \omega^i$, and so by Cartan's lemma we have

$$\omega_i^\alpha = \sum h_{ij}^\alpha \omega^j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{1}$$

where $i, j, \ldots = 1, \ldots, n$ and $\alpha, \beta, \ldots = n+1, \ldots, 2n$. Also $d\omega^i = -\Sigma\,\omega_j^i \wedge \omega^j$, $\omega_j^i + \omega_i^j = 0$. Since M is minimal, $\Sigma\, h_{ii}^\alpha = 0$ for all α. Also, if σ is the second fundamental form of the immersion, then $\|\sigma\|^2 = \Sigma\,(h_{ij}^\alpha)^2$. Let A_α denote the matrix formed from h_{ij}^α and $\operatorname{tr} A_\alpha$ denote the trace of A_α. Then, by [2, Proposition 3.5], we have

$$\tfrac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum \operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum(\operatorname{tr} A_\alpha A_\beta)^2 + (n+1)\|\sigma\|^2, \tag{2}$$

where Δ is the Laplacian operator and ∇' is covariant differentiation in (tangent bundle) $\oplus$ (normal bundle).

We have the following lemma from [3].

Lemma 1. *Let A and B be symmetric $(n \times n)$-matrices. Then* $-\operatorname{tr}(AB - BA)^2 \leq 2 \operatorname{tr} A^2 \operatorname{tr} B^2$, *and the equality holds for nonzero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where*

$$\tilde{A} = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & O \\ \hline \multicolumn{2}{c|}{O} & O \end{array}\right), \quad \tilde{B} = \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & O \\ \hline \multicolumn{2}{c|}{O} & O \end{array}\right).$$

Moreover, if A_1, A_2, A_3 are $(n \times n)$-symmetric matrices, and if

$$-\operatorname{tr}(A_a A_b - A_b A_a)^2 = 2 \operatorname{tr} A_a^2 \operatorname{tr}_b^2, \qquad 1 \leq a, b \leq 3,$$

then at least one of the matrices A_a must be zero.

Applying the inequality in the lemma to (2), we have

$$\begin{aligned} \tfrac{1}{2}\Delta\|\sigma\|^2 &\geq \|\nabla'\sigma\|^2 - 2 \sum_{\alpha\neq\beta} \operatorname{tr} A_\alpha^2 \operatorname{tr} A_\beta^2 - \sum(\operatorname{tr} A_\alpha A_\beta)^2 + (n+1)\|\sigma\|^2 \\ &= \|\nabla'\sigma\|^2 + (n+1)\|\sigma\|^2 - (2 - 1/n)\|\sigma\|^4 + n(n-1)(\sigma_1^2 - \sigma_2), \end{aligned} \tag{3}$$

where $n\sigma_1 = \Sigma \operatorname{tr} A_\alpha^2$, $n(n-1)\sigma_2 = 2\,\Sigma_{\alpha<\beta} \operatorname{tr} A_\alpha^2 \operatorname{tr} A_\beta^2$ and $n^2(n-1)(\sigma_1^2 - \sigma_2) = \Sigma_{\alpha<\beta}(\operatorname{tr} A_\alpha^2 - \operatorname{tr} A_\beta^2)^2$. Using the inequality in Theorem 1 and Hopf's Lemma, it is now easy to see the proof of Theorem 1. Assume now that $\|\sigma\|^2 \equiv (n+1)/(2 - 1/n)$. Then $\Delta\|\sigma\|^2 = 0$, and so by (3) we have

$$-\operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 = 2 \operatorname{tr} A_\alpha^2 \operatorname{tr} A_\beta^2, \qquad n+1 \leq \alpha \neq \beta \leq 2n, \tag{4}$$

(5) $$\|\nabla'\sigma\|^2 = 0,$$

and

(6) $$\operatorname{tr} A_\alpha^2 = \operatorname{tr} A_\beta^2, \qquad n+1 \leq \alpha \neq \beta \leq 2n.$$

By Lemma 1, (4) implies that at most two of the A_α's are nonzero. However, by (6), if one A_α is zero then all the A_α are zero and $\sigma \equiv 0$, which is not the case. Thus $n \leq 2$. If $n = 1$ then M is just a real curve in CP^1 and totally real is no restriction. Thus, we will assume $n = 2$ and by Lemma 1 we have

$$A_3 = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and, by (5), we see that λ and μ are constants. By (6) we have $\lambda^2 = \mu^2$, and since $\|\sigma\|^2 = 2$, we have $\lambda^2 + \mu^2 = 1$ so that $\lambda^2 = \frac{1}{2}$ and we may assume that $-\lambda = \mu = 1/\sqrt{2}$.

Now $\omega_i^\alpha = \Sigma h_{ij}^\alpha \omega^j$ so that

$$\omega_1^3 = \sum h_{1j}^3 \omega^j = \lambda\omega^2, \quad \omega_2^3 = \sum h_{2j}^3 \omega^j = \lambda\omega^1,$$

$$\omega_1^4 = \sum h_{1j}^4 \omega^j = \mu\omega^1, \quad \omega_2^4 = \sum h_{2j}^4 \omega^j = -\mu\omega^2.$$

If we differentiate (1) and use (5), we obtain (see [3])

$$0 = -\sum h_{il}^\alpha \omega_j^l - \sum h_{lj}^\alpha \omega_i^l + \sum h_{ij}^\beta \omega_\beta^\alpha.$$

Setting $\alpha = 3$ and $i = j = 1$, this becomes

$$0 = -\sum h_{1l}^3 \omega_1^l - \sum h_{l1}^3 \omega_1^l + \sum h_{11}^\beta \omega_\beta^3 = -\lambda\omega_1^2 - \lambda\omega_1^2 + \mu\omega_4^3.$$

That is $\omega_4^3 = 2\lambda/\mu\omega_1^2$. From formula (3.5) of [2] we see that M is a flat manifold. Putting this all together we have

Theorem 2. *Let M be an n-dimensional $(n > 1)$, minimal, totally real submanifold of CP^n satisfying $\|\sigma\|^2 = (n+1)/(2 - 1/n)$. Then $n = 2$ and, with respect to an adapted dual orthonormal frame field $\omega^1, \omega^2, \omega^3, \omega^4$, the connection form (ω_B^A) of CP^2 restricted to M is given by*

$$\begin{pmatrix} 0 & \omega_2^1 & -\lambda\omega^2 & -\mu\omega^1 \\ -\omega_2^1 & 0 & -\lambda\omega^1 & \mu\omega^2 \\ \lambda\omega^2 & \lambda\omega^1 & 0 & 2\omega_2^1 \\ \mu\omega^1 & -\mu\omega^2 & -2\omega_2^1 & 0 \end{pmatrix}, \qquad -\lambda = \mu = \frac{1}{\sqrt{2}}.$$

Therefore such a submanifold is locally unique. Thus, we have

Theorem 3. *If M is a compact n-dimensional $(n > 1)$, minimal, totally real submanifold of CP^n satisfying $\|\sigma\|^2 = (n+1)/(2 - 1/n)$, then $n = 2$ and $M = S^1 \times S^1$.*

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823

DEPARTMENT OF MATHEMATICS, SAITAMA UNIVERSITY, URAWA CITY, JAPAN

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO, JAPAN