TWO EXAMPLES IN PROXIMITY SPACES

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ABSTRACT. Two examples of Lo-spaces are given. The first is an example of a Lo-space in which not every ultrafilter is contained in a cluster. In the Lo-space of the second example, each ultrafilter is contained in a cluster, and yet not every maximal bunch is a cluster.

It is well known that in Efremovič proximity spaces each maximal bunch is a cluster and also each ultrafilter is contained in a unique cluster. In Example 1 we construct a Lo-space in which not every ultrafilter is contained in a cluster and, consequently, in that space, not every maximal bunch is a cluster. Surprisingly there also exist Lo-spaces in which every ultrafilter is contained in a cluster and still there are maximal bunches which are not clusters. One such space is outlined in Example 2.

We shall be using the terminology of [lj, some of which is given below.

Let \( \delta \) be a binary relation on the power set of a nonempty set \( X \). Consider the following axioms:

(P0) \( \{x, y\} \in \delta \) implies \( x = y \);
(P1) \( (\emptyset, x) \notin \delta \);
(P2) \( (A, B) \in \delta \) implies \( (B, A) \in \delta \);
(P3) \( A \cap B \neq \emptyset \) implies \( (A, B) \in \delta \);
(P4) \( (A, B \cup C) \in \delta \) if and only if \( (A, B) \in \delta \) or \( (A, C) \in \delta \);
(P5) \( (A, B) \in \delta \) and \( \{b\}, C \in \delta \) for each \( b \in B \) implies \( (A, C) \in \delta \);
(P6) \( (A, B) \notin \delta \) implies that there exists a subset \( E \) of \( X \) such that \( (A, E) \notin \delta \) and \( (X - E, B) \notin \delta \).

(i) \( \delta \) satisfying (P1-P5) is called a Lo-proximity.
(ii) \( \delta \) satisfying (P1-P4) and (P6) is called an Efremovič proximity (or EF proximity).
(iii) \( \delta \) satisfying (P0) is called separated.

Clearly every EF proximity is a Lo-proximity but not conversely. If \( \delta \) is a Lo-proximity (EF proximity) on \( X \), then the pair \( (X, \delta) \) is called a Lo-space (resp. EF space).

A topological space \( X \) is \( R_0 \) if and only if for each \( x \in X \) and each neighborhood \( G \) of \( X \), we have \( \{x\} \subseteq G \). A Lo-proximity \( \delta \) on a set \( X \) induces an \( R_0 \) topology on \( X \) via the Kuratowski closure operator given by

\[
\overline{A} = \{x \in X : \{x\}, A) \in \delta\}.
\]

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Let \((X, \delta)\) be a Lo-space. A collection \(\sigma\) of subsets of \(X\) is called a bunch provided \(\sigma\) is nonempty and satisfies the following conditions:

(a) \(A, B \in \sigma\) implies \((A, B) \in \delta\);

(b) \(A \cup B \in \sigma\) if and only if \(A \in \sigma\) or \(B \in \sigma\);

(c) \(\overline{A} \in \sigma\) implies \(A \in \sigma\).

\(\sigma\) is called a cluster if it is a bunch and satisfies the following:

(d) If \(G \subseteq X\) and \((G, A) \in \delta\) for each \(A \in \sigma\) then \(G \in \sigma\).

A full account of this and other related material is given in [1].

Example 1. This is an example of a Lo-space in which some ultrafilters are contained in no cluster. Since each ultrafilter is contained in a maximal bunch, the Lo-space constructed here will contain maximal bunches which are not clusters.

Let \(F_1, F_2, F_3\) and \(F_4\) be four distinct nonprincipal ultrafilters on an infinite set \(X\), and let \(F_5 = F_1\). Define a binary relation \(\delta\) on the power set of \(X\) as follows:

\[ A \delta B \text{ if and only if at least one of the following two conditions is satisfied:} \]

(i) \(A \cap B \neq \emptyset\);

(ii) For some \(i, 1 < i < 4\), one of the sets \(A, B\) is in \(F_i\) and the other belongs to \(F_{i+1}\).

It is easy to verify that \(\delta\) is a Lo-proximity. We claim that the filter \(F_i\) cannot be contained in any cluster for any \(i, 1 \leq i \leq 4\). We prove this for the filter \(F_1\). If possible suppose there exists a cluster \(\sigma\) such that \(F_1 \subseteq \sigma\). Then \(\overline{F_1} \subseteq \sigma \subseteq F_1 \cup F_2 \cup F_4\). Take any \(B \in F_2\) and \(C \in F_3\) such that \(C \notin F_1 \cup F_2 \cup F_4\). Then \(B \cup C \in F_2 \cap F_3\). Therefore \((B \cup C) \in \delta\) for each \(P \in F_1 \cup F_2 \cup F_2\). But since \(\sigma \subseteq F_1 \cup F_2 \cup F_4\), \((B \cup C) \in \delta\) for each \(P \in \sigma\). As \(\sigma\) is a cluster, we must have \(B \cup C \in \sigma\). But as \(C \notin F_1 \cup F_2 \cup F_4\), there exists \(A \in F_1\) such that \(C \notin A\). Consequently \(C \notin \sigma\), so \(B \notin \sigma\).

Since \(B \in F_2\) was arbitrary, we have \(F_2 \subseteq \sigma\). Similarly \(F_4 \subseteq \sigma\), and thus \(\sigma = F_1 \cup F_2 \cup F_4\). Now take two sets \(B\) and \(D\) such that \(B \cap D = \emptyset\), \(B \in F_2\), \(D \in F_4\), \(B \notin F_1 \cup F_3 \cup F_4\) and \(D \notin F_1 \cup F_2 \cup F_3\). By our choice \(B \notin D\) and both \(B\) and \(D\) belong to \(\sigma\). This is a contradiction. Thus we conclude that there is no cluster containing the filter \(F_1\). The same is true for the filters \(F_2, F_3\) and \(F_4\).

Example 2. This is an example of a Lo-space, in which, even though each ultrafilter is contained in a cluster, not every maximal bunch is a cluster.

Define a binary relation \(\delta\) on the power set of the set \(R\) of real numbers as follows:

\[ (A, B) \in \delta \text{ if and only if at least one of the following four conditions is satisfied} \]

(i) \(A \cap B \neq \emptyset\).
(ii) One of $A$ and $B$ contains an infinite subset of positive integers and the other contains an uncountable subset of positive real numbers.

(iii) One of $A$ and $B$ contains an infinite subset of negative integers and the other contains an uncountable subset of negative real numbers.

(iv) $A$ and $B$ are both uncountable.

The verification of $\delta$ being a separated Lo-proximity is straightforward. Also the collection $\zeta$ of all uncountable subsets of $R$ can easily be seen to be a bunch. We claim that $\zeta$ is a maximal bunch but not a cluster.

To show that $\zeta$ is a maximal bunch, take any bunch $\zeta_1$ such that $\zeta \subseteq \zeta_1$. It suffices to show that $\zeta = \zeta_1$. To see this let $A \in \zeta_1$. Write $A^+ = \{n \in A: n$ is a positive integer$\}$, $A^- = \{n \in A: n$ is a negative integer$\}$ and $B = \{x \in A: x \notin A^+ \cup A^-\}$. Since the set $R^-$ of all negative real numbers is in $\zeta$ and $(A^+, R^-) \notin \delta$, then $A^+ \notin \zeta$. Similarly $A^- \notin \zeta$, and therefore $A^+ \notin \zeta_1$ and $A^- \notin \zeta_1$. Since $A \in \zeta_1$ and $A = A^+ \cup A^- \cup B$, then $B \in \zeta_1$. Let $E$ be any uncountable subset of $R$. Then $E \in \zeta_1$ and therefore $(E, B) \in \delta$.

Since $B$ contains no positive integer nor any negative integers and $(E, B) \in \delta$ for any arbitrary uncountable subset $E$ of $R$, it follows from the definition of $\delta$ that $B$ is uncountable and, consequently, so is $A$. It follows that $A \in \zeta$ and, therefore, $\zeta = \zeta_1$. This proves that $\zeta$ is a maximal bunch. To show that $\zeta$ is not a cluster it is enough to observe that for the set $I$ of all integers we have $(I, A) \in \delta$ for each $A \in \zeta$, whereas $I \notin \zeta$.

Now we show that each ultrafilter on $(R, \delta)$ is contained in a cluster. Take any nonprincipal ultrafilter $\mathcal{F}$ on $R$. Then one of the sets $P = \{x \in R: x > 0\}$ and $N = \{x \in R: x < 0\}$ is in $\mathcal{F}$. Without any loss of generality, assume $P \in \mathcal{F}$. Let $I^+$ be the set of all positive integers. At least one of the following three cases holds.

Case I. $I^+ \in \mathcal{F}$. In this case the collection $\sigma = \{A \subseteq R: A \in \mathcal{F}$ or $A$ contains an uncountable subset of $P\}$ is a cluster containing $\mathcal{F}$.

Case II. $I^+ \notin \mathcal{F}$ and some member of $\mathcal{F}$ is countable. In this case $\mathcal{F}$ itself is a cluster.

Case III. Each member of $\mathcal{F}$ is uncountable. Let $\mathcal{G}$ be a nonprincipal ultrafilter on $I^+$. Then $\sigma = \{A \subseteq R: A \cap P$ is uncountable or $A$ contains some member of $\mathcal{G}\}$ is a cluster containing $\mathcal{F}$.

Thus in all cases, $\mathcal{F}$ is contained in a cluster.

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