RINGS WHOSE CYCLIC MODULES ARE INJECTIVE OR PROJECTIVE

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ABSTRACT. The object of this paper is to prove

Theorem. For a ring $R$ the following are equivalent:

(i) Every cyclic right $R$-module is injective or projective.
(ii) $R = S \oplus T$ where $S$ is semisimple artinian and $T$ is a simple right semihereditary right $\mathring{O}re$-domain whose every proper cyclic right module is injective.

Faith [2] called a ring $R$ a right PCI-ring if each cyclic right $R$-module $C \not\simeq R$ is injective, and has shown that a right PCI-ring is either semisimple artinian or else a simple right semihereditary right $\mathring{O}re$-domain. The proof of our Theorem provides an alternative and a shorter proof of Faith's result that if $R$ is a regular right PCI-ring (equivalently if $R$ is not a right PCI-domain), then $R$ is semisimple artinian. The existence of PCI-domains which are not division rings is given by Cozzens [1].

Throughout the lemmas the ring $R$ satisfies condition (i) of the Theorem.

Lemma 1. Let $I$ be a two-sided ideal of $R$. Then $R/I$ satisfies (i). Further, if $R/I$ is injective, then $R/I$ is semisimple artinian.

Proof. Clear by Osofsky [3].

Lemma 2. $R$ does not contain an infinite set of central orthogonal idempotents.

Proof. Let $\{e_i\}_{i \in \Lambda}$ be an infinite set of central orthogonal idempotents and $\Lambda'$ be an infinite subset of $\Lambda$ such that $\Lambda - \Lambda'$ is infinite. Set $A = \bigoplus_{i \in \Lambda} e_i R$, $B = \bigoplus_{i \in \Lambda - \Lambda'} e_i R$. $R/B$ cannot be projective since $B$ is infinitely generated. Hence $R/G$ is semisimple artinian. But $A/B$ is an infinitely generated ideal in $R/B$. This yields a contradiction.

Lemma 3. For any idempotent $e$ in $R$, $eR$ or $(1 - e)R$ is a completely reducible injective right $R$-module.

Proof. First we show that the right singular ideal $Z(R) = 0$.

If $a \in Z(R)$ then $aE = 0$ where $E$ is some essential right ideal of $R$. 

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This implies $aR \cong R/E$. If $aR$ is projective then $E = R$ and so $a = 0$. If $aR$ is injective then $aR = eR \subseteq Z(R)$, $e^2 = e$. This again implies $aR = 0$ and thus $a = 0$. Now write $R = eR \oplus (1 - e)R$. If both $eR$ and $(1 - e)R$ are completely reducible then the result is obvious. So suppose $eR$ is not completely reducible. Then there exists a proper essential $R$-submodule $A$ of $eR$.

$eR/A$ is then a singular $R$-module and since $Z(R) = (0)$, it cannot be projective. So from $R/A \cong eR/A \oplus (1 - e)R$, we get $R/A$ must be injective. Hence $(1 - e)R$ is injective. If $(1 - e)R$ were also not completely reducible, then, as before, we get $eR$ is injective, and hence $R$ is self-injective. Then Lemma 1 yields that $R$ is semisimple artinian and we are done. Thus in any case if $eR$ is not completely reducible, then $(1 - e)R$ is a completely reducible injective module.

**Lemma 4.** Either $R$ is an integral domain or $R$ has a nonzero socle.

**Proof.** By Lemma 3 if $R$ has an idempotent $e \neq 0, 1$, then $R$ has a nonzero socle. Now suppose $R$ dose not possess idempotents different from 0 and 1. Let $0 \neq a \in R$ and $r(a) = \{x \in R | ax = 0\}$. Then $aR \cong R/r(a)$. If $aR$ is injective then $aR = (0)$ or $aR = R$. The former implies $a = 0$ and the latter implies $R$ is right self-injective and hence semisimple artinian, consequently a division ring. If $aR$ is projective then $r(a) = (0)$ or $r(a) = R$. The latter is not possible. Hence $r(a) = 0$ and $R$ is an integral domain.

**Lemma 5.** If $R$ has no nontrivial central idempotents then either $R$ is simple artinian or $R$ is a simple right semihereditary right Ore-domain.

**Proof.** If $R$ is a domain then $R$ is a right PCI-ring and hence, by Faith [2, Propositions 5, 17], $R$ is a simple right semihereditary right Ore-domain. If $R$ is not a domain then $R$ has a nonzero socle $S$. From Lemma 3 every minimal right ideal of the form $eR$, $e = e^2 \in R$, is injective. Thus hypothesis (i) yields that every minimal right ideal $aR$ of $R$ is generated by an idempotent. But then it follows immediately that $R$ is semiprime. Indeed $R$ can be shown to be prime since $R$ has no nontrivial central idempotents. In case $R/S$ is projective, then $S = eR$ where $e$ is a central idempotent, so that $R = S$ is simple artinian. If $R/S$ is injective then $R/S$ is semisimple artinian, and hence $R$ is regular. Thus $R$ is a primitive regular ring with nonzero socle. Let $\hat{R}$ denote the maximal right quotient ring of $R$. Since every minimal right ideal is injective, the socle of $\hat{R} = \text{socle of } R = S$. If $S$ is finitely generated then $S = R = \hat{R}$ and hence $R$ is simple artinian. So assume that $S$ is not finitely generated. Then there exists right ideals $K_1, K_2$ in $S$ such that $S = K_1 \oplus K_2$ and $S \cong K_1 \approx K_2$. Since $K_1$ is infinitely generated, $R/K_1$ is injective. Also $R/K_1$ contains $(K_1 \oplus K_2)/K_1 \cong K_2 \approx S$. Hence $\hat{R}/K_1$ is embeddable in $R/K_1$. This implies $\hat{R} = xR$ for some $x \in \hat{R}$, so there exists $a \in R$ such that $xa = 1$. This implies $a + S$ is invertible in the semisimple
artinian ring $R/S$. Thus there exists $y + S$ in $R/S$ such that $ay - 1 \in S$. This yields that $x \in R$. Hence $R = \hat{R}$ which by Lemma 1 implies $R$ is simple artinian. This proves the lemma.

**Proof of the Theorem.** Assume (i). By Lemma 2 we can write $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ where $R_i$ are rings having no nontrivial central idempotents. Then by Lemma 3 we get all $R_i$, excepting at most one, are semisimple artinian. Hence either $R$ is semisimple artinian or $R = S \oplus T$ where $S$ is semisimple artinian and $T$ has no nontrivial central idempotents. Since $T$ also satisfies (i), Lemma 5 completes the proof. The converse is clear.

**REFERENCES**


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