PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 1, November 1975

RINGS WHOSE CYCLIC MODULES ARE INJECTIVE OR PROJECTIVE

S. C. GOEL, S. K. JAIN AND SURJETT SINGH

ABSTRACT. The object of this paper is to prove

Theorem. *For a ring R the following are equivalent:*

(i) *Every cyclic right R-module is injective or projective.*

(ii) *$R = S \oplus T$ where S is semisimple artinian and T is a simple right semihereditary right Öre-domain whose every proper cyclic right module is injective.*

Faith [2] called a ring R a right PCI-ring if each cyclic right R-module $C \not\approx R$ is injective, and has shown that a right PCI-ring is either semisimple artinian or else a simple right semihereditary right Öre-domain. The proof of our Theorem provides an alternative and a shorter proof of Faith's result that if R is a regular right PCI-ring (equivalently if R is not a right PCI-domain), then R is semisimple artinian. The existence of PCI-domains which are not division rings is given by Cozzens [1].

Throughout the lemmas the ring R satisfies condition (i) of the Theorem.

Lemma 1. *Let I be a two-sided ideal of R. Then R/I satisfies* (i). *Further, if R/I is injective, then R/I is semisimple artinian.*

Proof. Clear by Osofsky [3].

Lemma 2. *R does not contain an infinite set of central orthogonal idempotents.*

Proof. Let $\{e_i\}_{i\in\Lambda}$ be an infinite set of central orthogonal idempotents and Λ' be an infinite subset of Λ such that $\Lambda - \Lambda'$ is infinite. Set $A = \bigoplus\Sigma_{i\in\Lambda} e_i R$, $B = \bigoplus\Sigma_{i\in\Lambda-\Lambda'} e_i R$. R/B cannot be projective since B is infinitely generated. Hence R/G is semisimple artinian. But A/B is an infinitely generated ideal in R/B. This yields a contradiction.

Lemma 3. *For any idempotent e in R, eR or $(1-e)R$ is a completely reducible injective right R-module.*

Proof. First we show that the right singular ideal $Z(R) = 0$.

If $a \in Z(R)$ then $aE = 0$ where E is some essential right ideal of R.

Received by the editors July 2, 1974 and, in revised form, September 17, 1974.

AMS (MOS) subject classifications (1970). Primary 16A50, 16A52, 16A64; Secondary 16A12, 16A20.

Key words and phrases. Projective module, injective module, completely reducible module, self-injective ring, (von Neumann) regular ring, prime and semiprime rings, semisimple artinian ring, semihereditary Öre-domain.

This implies $aR \cong R/E$. If aR is projective then $E = R$ and so $a = 0$. If aR is injective then $aR = eR \subset Z(R)$, $e^2 = e$. This again implies $aR = 0$ and thus $a = 0$. Now write $R = eR \oplus (1-e)R$. If both eR and $(1-e)R$ are completely reducible then the result is obvious. So suppose eR is not completely reducible. Then there exists a proper essential R-submodule A of eR. eR/A is then a singular R-module and since $Z(R) = (0)$, it cannot be projective. So from $R/A \cong eR/A \oplus (1-e)R$, we get R/A must be injective. Hence $(1-e)R$ is injective. If $(1-e)R$ were also not completely reducible, then, as before, we get eR is injective, and hence R is self-injective. Then Lemma 1 yields that R is semisimple artinian and we are done. Thus in any case if eR is not completely reducible, then $(1-e)R$ is a completely reducible injective module.

Lemma 4. *Either R is an integral domain or R has a nonzero socle.*

Proof. By Lemma 3 if R has an idempotent $e \neq 0, 1$, then R has a nonzero socle. Now suppose R dose not possess idempotents different from 0 and 1. Let $0 \neq a \in R$ and $r(a) = \{x \in R \mid ax = 0\}$. Then $aR \cong R/r(a)$. If aR is injective then $aR = (0)$ or $aR = R$. The former implies $a = 0$ and the latter implies R is right self-injective and hence semisimple artinian, consequently a division ring. If aR is projective then $r(a) = (0)$ or $r(a) = R$. The latter is not possible. Hence $r(a) = 0$ and R is an integral domain.

Lemma 5. *If R has no nontrivial central idempotents then either R is simple artinian or R is a simple right semihereditary right Öre-domain.*

Proof. If R is a domain then R is a right PCI-ring and hence, by Faith [2, Propositions 5, 17], R is a simple right semihereditary right Öre-domain. If R is not a domain then R has a nonzero socle S. From Lemma 3 every minimal right ideal of the form eR, $e = e^2 \in R$, is injective. Thus hypothesis (i) yields that every minimal right ideal aR of R is generated by an idempotent. But then it follows immediately that R is semiprime. Indeed R can be shown to be prime since R has no nontrivial central idempotents. In case R/S is projective, then $S = eR$ where e is a central idempotent, so that $R = S$ is simple artinian. If R/S is injective then R/S is semisimple artinian, and hence R is regular. Thus R is a primitive regular ring with nonzero socle. Let $\hat{R}$ denote the maximal right quotient ring of R. Since every minimal right ideal is injective, the socle of $\hat{R}$ = socle of $R = S$. If S is finitely generated then $S = R = \hat{R}$ and hence R is simple artinian. So assume that S is not finitely generated. Then there exists right ideals K_1, K_2 in S such that $S = K_1 \oplus K_2$ and $S \approx K_1 \approx K_2$. Since K_1 is infinitely generated, R/K_1 is injective. Also R/K_1 contains $(K_1 \oplus K_2)/K_1 \approx K_2 \approx S$. Hence $\hat{R}_R = \hat{S}_R$ is embeddable in R/K_1. This implies $\hat{R} = xR$ for some $x \in \hat{R}$, so there exists $a \in R$ such that $xa = 1$. This implies $a + S$ is invertible in the semisimple artinian ring R/S. Thus there exists $y + S$ in R/S such that $ay - 1 \in S$. This yields that $x \in R$. Hence $R = \hat{R}$ which by Lemma 1 implies R is simple artinian. This proves the lemma.

Proof of the Theorem. Assume (i). By Lemma 2 we can write $R = R_1 \oplus R_2 \oplus \dots \oplus R_n$ where R_i are rings having no nontrivial central idempotents. Then by Lemma 3 we get all R_i, excepting at most one, are semisimple artinian. Hence either R is semisimple artinian or $R = S \oplus T$ where S is semisimple artinian and T has no nontrivial central idempotents. Since T also satisfies (i), Lemma 5 completes the proof. The converse is clear.

REFERENCES

1. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. Amer. Math. Soc. 76 (1970), 75–79. MR 41 #3531.

2. C. Faith, *When are proper cyclics injective?* Pacific J. Math. 45 (1973), 97–112. MR 47 #8610.

3. B. L. Osofsky, *Noninjective cyclic modules*, Proc. Amer. Math. Soc. 19 (1968), 1383–1384. MR 38 #185.

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701 (Current address of S. C. Goel and S. K. Jain)

Current address (Surjeet Singh): Department of Mathematics, Guru Nanak University, Amritsar (Panjab), India