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A COMMON FIXED-POINT THEOREM FOR COMPACT CONVEX SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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ABSTRACT. Let C be a bounded closed convex subset of a strictly convex Banach space and let S be a semigroup of nonexpansive self-mappings of C which is convex and compact in the topology of weak pointwise convergence. If S has the property that $\overline{\text{co}}\,\mathfrak{R}(s_1) \cap \overline{\text{co}}\,\mathfrak{R}(s_2) \neq \emptyset$ whenever $s_1, s_2 \in S$, then S has a common fixed point and $F(S)$ is a nonexpansive retract of C.

Throughout this paper, C denotes a bounded closed convex subset of a (real or complex) Banach space X. A family S of mappings $s: C \to C$ is a *semigroup* if it is closed under composition; S is *convex* if it is convex in the vector space X^C (with the usual pointwise operations). By a common fixed point of S we mean a point x such that $s(x) = x$ for all s in S; the set of common fixed points is denoted by $F(S)$. We give X^C the product topology after giving X its weak topology, so that compactness of S refers to its compactness in the topology of *weak* pointwise convergence. We say that S satisfies (FP), (F), (D+), (D), or (I), according to whether the following hold for every pair s_1, s_2 in S:

(FP): S has a common fixed point;

(F): s_1 and s_2 have a common fixed point;

(D+): $\mathfrak{R}(s_1) \cap \mathfrak{R}(s_2) \neq \emptyset$;

(D): $\text{dis}(\mathfrak{R}(s_1), \mathfrak{R}(s_2)) = 0$;

(I): $\overline{\text{co}}\,\mathfrak{R}(s_1) \cap \overline{\text{co}}\,\mathfrak{R}(s_2) \neq \emptyset$,

where $\mathfrak{R}(s)$ denotes the range of s and $\overline{\text{co}}$ denotes convex closure. Evidently (FP) $\Rightarrow$ (F) $\Rightarrow$ (D+) $\Rightarrow$ (D) and, if C is weakly compact, (D) $\Rightarrow$ (I). Evidently, too, the nature of conditions (D+), (D), and (I) is different from the nature of (FP) and (F): the former are nonseparation assumptions on the *ranges* of mappings in S, and do not directly refer to fixed points. Nevertheless, our main result is that (I) $\Rightarrow$ (FP) if X is strictly convex and the mappings in S are nonexpansive. Indeed, $F(S)$ is then a nonexpansive retract of C—the range of a nonexpansive retraction. (For properties of nonexpansive retracts, see [1], [2], [3].)

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Theorem 1. *If X is strictly convex and S is a compact, convex semigroup of nonexpansive self-mappings of C which satisfies* (I), *then $F(S)$ is a nonempty nonexpansive retract of C.*

Proof. Define a partial order $\leq$ on S by setting $f < g$ to mean $\|fx - fy\| \leq \|gx - gy\|$ for all x, y in C, with inequality holding for at least one pair x, y, and $f \leq g$ to mean $f < g$ or $f = g$. This order was introduced in [2], [3]. As in the proof of [3, Lemma 2], there exists a minimal element r in $(S, \leq)$, and each s in S acts as an isometry on $\mathcal{R}(r)$:

$$\|sr(x) - sr(y)\| = \|r(x) - r(y)\|. \tag{1}$$

[The proof of Lemma 2 in [3] is inaccurate in that the initial segments $\mathrm{Is}(g) = \{f \in S: f \leq g\}$ are not compact, as claimed. However, if $g_1 < g_2$ then $\mathrm{cl}\ \mathrm{Is}(g_1)$ is compact and is contained in $\mathrm{Is}(g_2)$, and this is all that is needed to prove the existence of a minimal r.]

If r is minimal in $(S, \leq)$ and $s \in S$, then $\frac{1}{2}sr + \frac{1}{2}r \in S$ and

$$\begin{aligned} &\|(\tfrac{1}{2}sr + \tfrac{1}{2}r)(x) - (\tfrac{1}{2}sr + \tfrac{1}{2}r)(y)\| \\ &\qquad \leq \tfrac{1}{2}\|sr(x) - sr(y)\| + \tfrac{1}{2}\|r(x) - r(y)\| \leq \|r(x) - r(y)\|. \end{aligned} \tag{2}$$

Equality must hold throughout (2) since r is minimal, hence, by (1) and the strict convexity of X, $sr(x) - sr(y) = r(x) - r(y)$. Rephrased, if r is minimal in S, then each s in S acts as a translation on $\mathcal{R}(r)$.

In particular, r acts as a translation on $\mathcal{R}(r)$. But $\mathcal{R}(r)$ is bounded and r-invariant, so this means r acts as the identity on $\mathcal{R}(r)$. Thus r is a (nonexpansive) retraction of C onto $\mathcal{R}(r)$.

Let r_1, r_2 be a minimal in $(S, \leq)$. We claim $\mathcal{R}(r_1) = \mathcal{R}(r_2)$. Indeed, we have already shown that r_1 acts as a translation by some vector v on $\mathcal{R}(r_2)$ and as the identity on $\mathcal{R}(r_1)$; but $\mathcal{R}(r_1)$ and $\mathcal{R}(r_2)$ are closed and convex (they are the fixed-point sets of the nonexpansive mappings r_1 and r_2, and X is strictly convex; see [6]); so condition (I) implies $\mathcal{R}(r_1) \cap \mathcal{R}(r_2) \neq \emptyset$. Thus $v = 0$. That is, r_1 acts as the identity on $\mathcal{R}(r_2)$, so that $\mathcal{R}(r_2) \subset \mathcal{R}(r_1)$. By symmetry, $\mathcal{R}(r_1) = \mathcal{R}(r_2)$ as claimed.

Next, we claim that if r is minimal in $(S, \leq)$ then $\mathcal{R}(r) = F(S)$. Obviously $F(S) \subset \mathcal{R}(r)$. To prove the reverse inclusion, let $s \in S$. By virtue of (1), sr is also minimal in $(S, \leq)$. But we have shown that minimal elements of S are retractions, all of which have the same range; therefore sr is a retraction of C onto $\mathcal{R}(r)$. If $x \in \mathcal{R}(r)$ then $r(x) = x$ and $sr(x) = x$; so $s(x) = x$. Since this is true for all s in S, we have proven $\mathcal{R}(r) \subset F(S)$, and hence $\mathcal{R}(r) = F(S)$.

$F(S)$ is nonempty because, obviously, $\mathcal{R}(r) \neq \emptyset$, and r is a nonexpansive retraction of C onto $F(S)$. Q.E.D.

In practice, the most onerous assumption in Theorem 1 is that S is compact in the topology of weak pointwise convergence. It is usually fairly easy to generate convex semigroups which satisfy (I). For example, suppose T: $C \to C$ is nonexpansive. The existence of a sequence $\{x_n\}$ such that $\lim_n \|x_n - Tx_n\| = 0$ is standard; put $S = \{s\colon s$ is an nonexpansive self-mapping of C and $\lim_n \|x_n - s(x_n)\| = 0\}$. Obviously S is convex and satisfies (D); hence, if C is also weakly compact, S satisfies (I). S is a semigroup because

$$\|s_1 s_2(x) - x\| \leq \|s_1 s_2(x) - s_1(x)\| + \|s_1(x) - x\| \leq \|s_2(x) - x\| + \|s_1(x) - x\|$$

whenever s_1 is nonexpansive. Evidently $T \in S$, so that a common fixed point of S is a fixed point of T. But we are unable to use Theorem 1 to prove the existence of a fixed point of T because apparently S may not be compact.

The situation is different when C is strongly compact.

Theorem 2. *If X is strictly convex, C is strongly compact, and S is merely a convex semigroup of nonexpansive self-mappings of C which satisfies* (I), *then S also satisfies* (FP).

Proof. Since C is compact and S is equicontinuous, the closure $\bar{S}$ of S in C^C is also the closure of S in the topology of uniform convergence, and the weak pointwise convergence topology on $\bar{S}$ is the same as the topology of uniform convergence [5, p. 232]. Obviously S and $\bar{S}$ have the same fixed points. Since mappings in $\bar{S}$ can be uniformly approximated by mappings in S, it is easy to see that $\bar{S}$ satisfies (I) if S does. By Theorem 1, therefore, $\bar{S}$ satisfies (FP), hence so does S. Q.E.D.

Example. (I) does not imply (FP) if X is not strictly convex, even if C is compact. We give an example patterned after DeMarr [4]. Let X be R^2 with the sup norm and let C be the square $\{(x, y)\colon |x| \leq 1, |y| \leq 1\}$. For $0 \leq t \leq 1$ define $f_t(x, y) = (|y| - t, y)$, and put $S = \{f_t\colon 0 \leq t \leq 1\}$. Since $f_t f_s = f_t$ and $\lambda f_t + (1 - \lambda) f_s = f_{\lambda t + (1-\lambda)s}$, S is a convex semigroup. Evidently S is compact and each f_t in S is nonexpansive. (I) is satisfied because the range of f_t is the broken line segment joining $(1 - t, 1)$ to $(-t, 0)$ to $(1 - t, -1)$, so that $(0, 0) \in \bigcap\{\overline{\mathrm{co}}\, \mathcal{R}(f_t)\colon 0 \leq t \leq 1\}$. Nevertheless, none of the conditions (FP), (F), (D+), or (D) is satisfied.

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