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A NOTE ON A DIFFERENTIAL CONCOMITANT

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ABSTRACT. If $\mathbf{h}$ and $\mathbf{k}$ are vector 1-forms, the vanishing of the concomitant $[\mathbf{h}, \mathbf{k}]$ is an integrability condition for certain problems on manifolds. In the case that $\mathbf{h} = \mathbf{k}$ the vanishing of the Nijenhuis tensor $[\mathbf{h}, \mathbf{h}]$ implies $d(\operatorname{tr} \mathbf{h})$ is a conservation law for $\mathbf{h}$, provided that $\operatorname{tr} \mathbf{h}$ is not constant. When the trace of $\mathbf{h}$ is constant, a conservation law for $\mathbf{h}$ exists if one can find a vector 1-form $\mathbf{k}$ with nonconstant trace such that $[\mathbf{h}, \mathbf{k}] = 0$.

1. **Introduction.** For any vector 1-form $\mathbf{h}$ there is an associated operator d_h which is an antiderivation of degree one. In the case that $\mathbf{h}$ is nonsingular and has vanishing Nijenhuis tensor $[\mathbf{h}, \mathbf{h}]$, the operator d_h satisfies a Poincaré lemma and thus provides an $\mathbf{h}$-dependent version of de Rham's theorem. Such an operator is an example of a special type of derivation whose basic properties were studied in a definitive paper by A. Frölicher and A. Nijenhuis [3]. When $\mathbf{h}$ is the identity on 1-forms, the operator d_h reduces to the usual exterior differentiation operator d. This derivation d_h and other operators related to it may be used to give an alternative description of the differential concomitant $[\mathbf{h}, \mathbf{k}]$ of vector 1-forms $\mathbf{h}$ and $\mathbf{k}$. The vanishing of this concomitant yields the main result of this paper. It is an identity which simply says that the composition ($d \circ \operatorname{trace}$) acts as a first order derivation on the composition $\mathbf{hk}$.

The derivational property of $d \circ \operatorname{trace}$ is applicable in the study of conservation laws. With it one can establish the existence of conservation laws in a constructive manner which has the added advantage of giving a global result. Since conservation laws have usually been mathematically studied in only a local manner, global results are few. For example, in general relativity the effect of a global conservation law (Einstein's equations) on the manifold topology is of current physical interest (see [2]). Implicit in any global conservation law problem is an interaction with topology; the derivational identity yields a possible method which can be applied in such global problems.

2. **Preliminaries.** Let A denote the algebra of C^∞ functions on a compact, orientable, n-dimensional Riemannian manifold M without boundary.

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Let E denote the A-module of differential 1-forms on M. A vector 1-form $h \in \mathrm{End}_A E$ induces endomorphisms $h^{(q)} \in \mathrm{End}_A(\bigwedge^p E)$ for any nonnegative integer q, and the $h^{(q)}$ are defined by setting $h^{(q)} = 0$ if $q > p$ and

$$h^{(q)}(\phi^1 \wedge \cdots \wedge \phi^p) = \frac{1}{(p-q)!q!} \sum_{\pi} |\pi| \{\mathbf{h}\phi^{\pi(1)} \wedge \cdots \wedge \mathbf{h}\phi^{\pi(q)}\}$$

$$\wedge \phi^{\pi(q+1)} \wedge \cdots \wedge \phi^{\pi(p)}$$

if $0 \leq q \leq p$, where $\phi^i \in E$, π runs through all permutations of $(1, \ldots, p)$ and $|\pi|$ denotes the sign of the permutation. The transformation $h^{(0)}$ is taken to be the identity mapping on $\bigwedge^p E$.

In the case where $q = p \leq n$, the operator $h^{(p)}$ is locally represented by an $\binom{n}{p} \times \binom{n}{p}$ matrix $[h^{(p)}]$ relative to some local basis of p-forms. If $[\mathbf{h}]$ denotes an $n \times n$ matrix which locally represents $\mathbf{h}$, then it can be shown that

$$\det[h^{(p)}] = (\det[\mathbf{h}])^{\binom{n-1}{p-1}}$$

and hence $\mathbf{h}$ is invertible on 1-forms if and only if $h^{(p)}$ is invertible on p-forms.

An alternating derivation $d_h: \bigwedge E \to \bigwedge E$ is obtained from $\mathbf{h}$ and an exterior derivation d by setting $d_h = h^{(1)}d - dh^{(1)}$. Thus when $\mathbf{h}$ is the identity, d_h reduces to d. The Nijenhuis tensor can be extended as a derivation on $\bigwedge E$. On p-forms one thus obtains the formula

$$[\mathbf{h}, \mathbf{h}] = -h^{(2)}d + d_h h^{(1)} + dh^{(2)},$$

which can be rewritten in the form

$$[\mathbf{h}, \mathbf{h}] = \tfrac{1}{2}\{d_{h^2} - (d_h)_h\}.$$

In a similar way the concomitant $[\mathbf{h}, \mathbf{k}]$ of vector 1-forms defined by the equation

$$[\mathbf{h}, \mathbf{k}] = \tfrac{1}{2}\{-[h^{(1)}k^{(1)} - (hk)^{(1)}]d$$

$$+ [h^{(1)}dk^{(1)} + k^{(1)}dh^{(1)}] - [d[(hk)^{(1)} + k^{(1)}h^{(1)}]]\}$$

can be rewritten in the form

$$[\mathbf{h}, \mathbf{k}] = \tfrac{1}{2}\{d_{hk} - (d_h)_k\}.$$

3. **The main identity.** Let $\omega^1, \ldots, \omega^n$ be a local orthonormal basis of differential 1-forms and suppose that $\omega^1 \wedge \cdots \wedge \omega^n$ agrees with an orientation of M. Suppose also that the vector 1-form $\mathbf{h}$ is locally specified by setting $\mathbf{h}\omega^j = h^j_i \omega^i$, where $h^j_i \in A$ and $1 \leq i, j \leq n$, and the Einstein summation convention has been invoked. Since the orientation is specified, the Hodge star operator $*: \bigwedge^p E \to \bigwedge^{n-p} E$ is determined by setting

$$*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = \epsilon_{i_1 \cdots i_p} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}}$$

where $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_{n-p}$ are complementary sets of positive integers and $\epsilon_{i_1 \cdots i_p}$ is $+1$ or -1 if the permutation $(i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$ of the integers $(1, 2, \ldots, n)$ is even or odd respectively. In the sequel the trace of $\mathbf{h}$ and the transpose of $\mathbf{h}$ will be denoted by $\operatorname{tr} \mathbf{h}$ and $\mathbf{h}_t$ respectively. The following lemma is a key to establishing our identity. The proof of the lemma appears in [1].

Lemma 3.1. *For any vector 1-form* $\mathbf{h}$,

$$h^{(1)}* + *h_t^{(1)} = (\operatorname{tr} \mathbf{h})*.$$

Since the manifold M is assumed to be compact, an inner product on $\bigwedge^p E$ is defined by setting

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta$$

for any p-forms α and β. With respect to this inner product the adjoint of $h^{(p)}$ is $h_t^{(p)}$ for $p = 0, 1, \ldots, n$.

The codifferential δ is defined to be the adjoint of the exterior derivative d. The adjoint of $d_h = h^{(1)}d - dh^{(1)}$ is then easily seen to be $\delta_h = \delta h_t^{(1)} - h_t^{(1)}\delta$. The operator δ_h is called the $\mathbf{h}$-codifferential. As a consequence of Stokes' theorem one obtains the relation $\delta = (-1)^{np+n+1} * d *$ on p-forms. The corresponding expression for δ_h is obtained in the following lemma.

Lemma 3.2. *If* $\mathbf{h}$ *is a vector* 1*-form, then on p-forms*

$$\delta_h = (-1)^{np+n+1} *\{d_h + d(\operatorname{tr} \mathbf{h}) \wedge\}*.$$

Proof. On p-forms one obtains

$$\delta_h = \delta h_t^{(1)} - h_t^{(1)}\delta = (-1)^{np+n+1}\{*d*h_t^{(1)} - h_t^{(1)}*d*\},$$

and thus Lemma 3.1 may then be applied to yield

$$\begin{aligned}\delta_h &= (-1)^{np+n+1}\{*d[\operatorname{tr} \mathbf{h}* - h^{(1)}*] - [*\operatorname{tr} \mathbf{h} - *h^{(1)}]d*\} \\ &= (-1)^{np+n+1}\{*d(\operatorname{tr} \mathbf{h})* - *dh^{(1)}* + *h^{(1)}d*\} \\ &= (-1)^{np+n+1} *\{d_h + d(\operatorname{tr} \mathbf{h}) \wedge\}*.\end{aligned}$$

The next lemma establishes a formula for $\operatorname{adj}[\mathbf{h}, \mathbf{k}]$ for any $\mathbf{h}, \mathbf{k} \in \operatorname{End}_A E$; this formula is analogous to the formula for δ_h in the preceding lemma.

Lemma 3.3. *For any* $\mathbf{h}, \mathbf{k} \in \operatorname{End}_A E$,

$$\operatorname{adj}[\mathbf{h}, \mathbf{k}] = (-1)^{np+1} *\{[\mathbf{h}, \mathbf{k}] + \tfrac{1}{2}\{d(\operatorname{tr} \mathbf{hk}) - \mathbf{h}d(\operatorname{tr} \mathbf{k}) - \mathbf{k}d(\operatorname{tr} \mathbf{h})\} \wedge \cdot\}*$$

on p-forms.

Proof. Since

$$*[\mathbf{h}, \mathbf{k}] = \frac{1}{2} *\{d_{hk} - (d_h)_k\} = \frac{(-1)^{p(n-p)}}{2}\{*d_{hk}* - *(d_h)_k*\}*$$

on p-forms, it suffices to compute $*d_{hk}*$ and $*(d_h)_k*$ on $(n-p)$ 1-forms. Thus, as a consequence of Lemma 3.2, one has

$$*d_{hk}* = (-1)^{np+1}\delta_{hk} - *d(\operatorname{tr} \mathbf{hk})*$$

and

$$*(d_h)_k* = *\{k^{(1)}d_h - d_h k^{(1)}\}*$$

$$= \{\operatorname{tr} \mathbf{k} - k_t^{(1)}\}*d_h* - *d_h*\{\operatorname{tr} \mathbf{k} - k_t^{(1)}\}$$

by Lemma 3.1. The expression for $*(d_h)_k*$ then reduces to

$$*(d_h)_k* = (-1)^{np+1}(\delta_h)_k - *\{\mathbf{k}d(\operatorname{tr} \mathbf{h})* + \mathbf{h}d(\operatorname{tr} \mathbf{k})\}*$$

and, hence,

$$*[\mathbf{h}, \mathbf{k}] = (-1)^{p+1}\tfrac{1}{2}\{\delta_{hk} - (\delta_h)_k\}*$$

$$+ *\tfrac{1}{2}\{\mathbf{k}d(\operatorname{tr} \mathbf{h}) + \mathbf{h}d(\operatorname{tr} \mathbf{k}) - d(\operatorname{tr} \mathbf{hk})\}\wedge\cdot.$$

Thus

$$*[\mathbf{h}, \mathbf{k}] = (-1)^{p+1}\operatorname{adj}[h, k]* + \tfrac{1}{2} *[\mathbf{k}d(\operatorname{tr} \mathbf{h}) + \mathbf{h}d(\operatorname{tr} \mathbf{k}) - d(\operatorname{tr} \mathbf{hk})\}\wedge\cdot$$

and the desired result is then obtained by an application of the Hodge star operator $*$, which is an isomorphism, on the right side of the above expression.

If $[\mathbf{h}, \mathbf{k}]$ vanishes, then Lemma 3.3 yields an identity which is the main result of this paper. The identity is given in the following proposition.

Proposition 3.4. *If* $[\mathbf{h}, \mathbf{k}] = 0$, *then* $d(\operatorname{tr} \mathbf{hk}) = \mathbf{h}d(\operatorname{tr} \mathbf{k}) + \mathbf{k}d(\operatorname{tr} \mathbf{h})$. *Thus* $D = d \circ \operatorname{trace}: \operatorname{End}_A E \longrightarrow E$ *is a first order derivation on compositions of vector* 1*-forms* $\mathbf{h}$ *and* $\mathbf{k}$ *whose differential concomitant* $[\mathbf{h}, \mathbf{k}]$ *vanishes.*

One can then study equations of the form $D\mathbf{h} = \alpha$ for some $\alpha \in E$ where a solution $\mathbf{h} \in \operatorname{End}_A E$ is desired. This equation is analogous to first order differential equations on the real line. The study of such equations will not be pursued here. However, the fact that D acts analogously to first derivatives on polynomial functions of one real variable will be shown in the following corollaries to Proposition 3.4.

Corollary 3.5. *If* $[\mathbf{h}, \mathbf{h}] = 0$, *then for any nonnegative integers* i *and* j,

$$\mathbf{h}^i d(\operatorname{tr} \mathbf{h}^j) + \mathbf{h}^j d(\operatorname{tr} \mathbf{h}^i) = d(\operatorname{tr} \mathbf{h}^{i+j}).$$

Proof. The vanishing of $[\mathbf{h}, \mathbf{h}]$ implies the vanishing of the concomitant $[\mathbf{h}^i, \mathbf{h}^j]$ for any nonnegative integers i and j. Thus, as a consequence of Proposition 3.4 with $\mathbf{h}^i$ and $\mathbf{h}^j$ in place of $\mathbf{h}$ and $\mathbf{k}$, the corollary is established.

Corollary 3.6. *If* $[\mathbf{h}, \mathbf{h}] = 0$, *then for any nonnegative integer* i, $\mathbf{h}^i d(\operatorname{tr} \mathbf{h}) = (i+1)^{-1} d(\operatorname{tr} \mathbf{h}^{i+1})$.

Proof. If $i = 1$, then $\mathbf{h}d(\operatorname{tr} \mathbf{h}) = \frac{1}{2}d(\operatorname{tr} \mathbf{h}^2)$ as a consequence of Corollary 3.5. If $\mathbf{h}^j d(\operatorname{tr} \mathbf{h}) = (j+1)^{-1} d(\operatorname{tr} \mathbf{h}^{j+1})$ for some positive integer $j > 1$, then

$$\mathbf{h}^{j+1} d(\operatorname{tr} \mathbf{h}) = \frac{1}{j+1} \mathbf{h} d(\operatorname{tr} \mathbf{h}^{j+1}) = \frac{1}{j+1} \{-\mathbf{h}^{j+1} d(\operatorname{tr} \mathbf{h}) + d(\operatorname{tr} \mathbf{h}^{j+2})\}$$

as another consequence of Corollary 3.5. Hence

$$\mathbf{h}^{j+1} d(\operatorname{tr} \mathbf{h}) = (j+2)^{-1} d(\operatorname{tr} \mathbf{h}^{j+2}).$$

This induction argument establishes the corollary.

Corollary 3.7. *If* $[\mathbf{h}, \mathbf{h}] = 0$, *and if* $\mathbf{h}$ *has constant trace, then the coefficients in the characteristic polynomial of* $\mathbf{h}$ *are constant.*

Corollary 3.7 is an immediate consequence of the fact that if $\mathbf{h}$ has constant trace, then all the positive powers of $\mathbf{h}$ also have constant trace.

In the case that the coefficients of the characteristic polynomial of a vector 1-form $\mathbf{h}$ are all constant, a result analogous to the formula $\delta * = (-1)^{p+1} * d$ on p-forms is obtained. Thus we have

Corollary 3.8. *If the coefficients of the characteristic polynomial of* $\mathbf{h}$ *are constant, then*

$$*[\mathbf{h}, \mathbf{h}] = (-1)^{p+1} (\operatorname{adj}[\mathbf{h}, \mathbf{h}]) *.$$

Proof. Let $\mathbf{h}^n = a_0 I + a_1 \mathbf{h} + \cdots + a_{n-1} \mathbf{h}^{n-1}$ be the characteristic equation of $\mathbf{h}$. Since $a_{n-2} = \frac{1}{2}\{\operatorname{tr}(\mathbf{h}^2) - a_{n-1}^2\}$, and $\operatorname{tr} \mathbf{h} = a_{n-1}$, it follows that $\operatorname{tr}(\mathbf{h}^2)$ is constant and, hence, $\mathbf{h}d(\operatorname{tr} \mathbf{h}) = \frac{1}{2}d(\operatorname{tr} \mathbf{h}^2) = 0$. Thus $*[\mathbf{h}, \mathbf{h}] = (-1)^{p+1} \operatorname{adj}[\mathbf{h}, \mathbf{h}] *$ on p-forms as a consequence of Lemma 3.3.

This section is concluded with an alternate version of the identity in Corollary 3.6. Since the traces of the various powers of $\mathbf{h}$ are expressible in terms of the coefficients a_i, the following proposition may be obtained.

Proposition 3.9. *If* $[\mathbf{h}, \mathbf{h}] = 0$ *and* $\mathbf{h}^n = a_0 I + a_1 \mathbf{h} + \cdots + a_{n-1} \mathbf{h}^{n-1}$ *is the characteristic equation of* $\mathbf{h}$, *then*

$$\mathbf{h} da_i = da_{i-1} + a_i da_n, \qquad i = 0, 1, \ldots, n-1 \ (da_{-1} \equiv 0).$$

Proof. For the case that $\mathbf{h}$ is cyclic, a proof appears in [6]. Otherwise the identity in Corollary 3.6 may be used. Thus $\mathbf{h}d(\operatorname{tr} \mathbf{h}) = \frac{1}{2}d(\operatorname{tr} \mathbf{h}^2)$ and, since $\operatorname{tr} \mathbf{h} = a_{n-1}$ and $\operatorname{tr} \mathbf{h}^2 = 2a_{n-2} + a_{n-1}^2$, one obtains $\mathbf{h} da_{n-1} = da_{n-2} + a_{n-1} da_{n-1}$. Hence, the case $i = n-1$ in the proposition is established. If $\mathbf{h}$ is applied to the above result, the relation obtained is

$$\begin{aligned} \mathbf{h} da_{n-2} &= \mathbf{h}^2 da_{n-1} - a_{n-1} \mathbf{h} da_{n-1} \\ &= \mathbf{h}^2 d(\operatorname{tr} \mathbf{h}) - a_{n-1} \mathbf{h} d(\operatorname{tr} \mathbf{h}) = (1/3) d(\operatorname{tr} \mathbf{h}^3) - (a_{n-1}/2) d(\operatorname{tr} \mathbf{h}^2); \end{aligned}$$

since $\operatorname{tr} \mathbf{h}^3 = 3a_{n-3} + 3a_{n-1}a_{n-2} + a_{n-1}^3$, the case $i = n - 2$ is established. Thus

$$\mathbf{h}da_{n-2} = da_{n-3} + a_{n-2}da_{n-1}.$$

The remaining identities can be established by repeated use of Corollary 3.6, though the calculation is tedious because of the form of the expressions for $\operatorname{tr} \mathbf{h}^i$.

Corollary 3.10. *If the Nijenhuis tensor* $[\mathbf{h}, \mathbf{h}]$ *vanishes identically, then the coefficients* a_i *in the characteristic equation for* $\mathbf{h}$ *are functionally independent if and only* $\mathbf{h}$ *is cyclic with generator* $d(\operatorname{tr} \mathbf{h})$.

Proof. It is sufficient to establish the relation

$$d(\operatorname{tr} \mathbf{h}) \wedge \mathbf{h}d(\operatorname{tr} \mathbf{h}) \wedge \cdots \wedge \mathbf{h}^{n-1}d(\operatorname{tr} \mathbf{h}) = da_{n-1} \wedge da_{n-2} \wedge \cdots \wedge da_0,$$

which is a consequence of the proposition.

4. **Conservation laws.** The conservation laws of physics can be expressed in conservation law form as a differential equation [4], and also as a relation between vector 1-forms and differential forms [5]. The use of vector 1-forms and differential forms trivially extends the problem to a global one on manifolds. Global conservation laws are of interest in physics as is evident from [2], but to date do not appear to have been studied in a general mathematical context. With the identity stated in Corollary 3.4 the global conservation law problem can be studied.

On a differentiable manifold a conservation law for a vector 1-form $\mathbf{h}$ is defined as any exact differential form θ such that $h\theta$ is also exact. Thus if $\theta = df$ for some differentiable function f, then θ is a conservation law for $\mathbf{h}$ if there is a differentiable function g such that $\mathbf{h}df = dg$. The analysis may be divided into two cases insofar as the existence of conservation laws is concerned. The cases correspond to constant and nonconstant trace for $\mathbf{h}$.

If $\mathbf{h}$ has trace which is not constant, then Corollary 3.6 says that $d(\operatorname{tr} \mathbf{h})$ is a conservation law for any positive power of $\mathbf{h}$. Moreover if $\mathbf{h}$ is cyclic with $d(\operatorname{tr} \mathbf{h})$ as a generator, then Corollary 3.1 implies that a basis of conservation laws can be obtained. This discussion is summarized in the following statement.

Proposition 4.1. *Let* M *be a compact orientable* n*-dimensional Riemannian manifold without boundary and let* $\mathbf{h}$ *be a vector* 1*-form whose trace is not constant on* M. *If the concomitant* $[\mathbf{h}, \mathbf{h}]$ *vanishes on* M, *then* $d(\operatorname{tr} \mathbf{h}^i)$ *with* $i = 1, 2, \ldots$ *is a conservation law for* $\mathbf{h}$, *and for fixed* i, $d(\operatorname{tr} \mathbf{h}^i)$ *is a conservation law for* $\mathbf{h}^j$, *with* $j = 1, 2, \ldots$. *Finally if* $\mathbf{h}$ *is cyclic with generator* $d(\operatorname{tr} \mathbf{h})$, *then the collection* $d(\operatorname{tr} \mathbf{h}^i)$, $i = 1, 2, \ldots, n$, *is a basis of conservation laws for* $\mathbf{h}$.

The analysis for the case in which $\operatorname{tr} \mathbf{h}$ is constant is not as simple.

The above reasoning can be extended provided that a nontrivial solution can be obtained for a certain system of differential equations. The solution of global differential equations is, of course, limited by topological considerations. In particular, if it is possible to find another vector 1-form $\mathbf{k}$ with nonconstant trace such that the concomitant $[\mathbf{h}, \mathbf{k}]$ vanishes, then the identity of Proposition 3.4 yields the statement that $d(\operatorname{tr} \mathbf{h})$ is a conservation law for $\mathbf{h}$. In local coordinates the vanishing of $[\mathbf{h}, \mathbf{k}]$ leads to a system of $n\binom{n}{2}$ equations (corresponding to n basis elements and $\binom{n}{2}$ elements in $E \wedge E$) in n^2 unknowns. This system may be written out explicitly if we assume that $(dx^1, \ldots, dx^n)$ is a local basis of 1-forms, that $\mathbf{h}dx^i = h^i_\alpha dx^\alpha$, $\mathbf{k}dx^i = k^i_\alpha dx^\alpha$, and that $(\ )_{,i} = \partial(\ )/\partial x^i$. Thus the condition $[\mathbf{h}, \mathbf{k}] = 0$ leads to a local system of the form

$$2(h^\alpha_j \delta^\beta_l - h^\beta_l \delta^\alpha_j) k^i_{\beta,\alpha}$$
$$= (2h^\alpha_{l,j}\delta^i_\beta + h^i_{j,\beta}\delta^\alpha_l + h^i_{\beta,l}\delta^\alpha_j - 2h^\alpha_{j,l}\delta^i_\beta - h^i_{l,\beta}\delta^\alpha_j - h^i_{s,j}\delta^\alpha_l)k^\beta_\alpha$$

for $1 \leq j < l \leq n$ and $1 \leq i \leq n$. For $n \geq 3$ the system $[\mathbf{h}, \mathbf{k}] = 0$ is generally overdetermined but may still have nontrivial solutions. To summarize this section one has

Proposition 4.2. *Let M be a compact orientable n-dimensional Riemannian manifold without boundary and let $\mathbf{h}$ be a vector 1-form on M. If there is a vector 1-form $\mathbf{k}$ with nonconstant trace and vanishing concomitant $[\mathbf{h}, \mathbf{k}]$, then $d(\operatorname{tr} \mathbf{k})$ is a conservation law for $\mathbf{h}$, provided $\operatorname{tr} \mathbf{h}$ is constant.*

REFERENCES

1. P. R. Eiseman and A. P. Stone, *A generalized Hodge theory*, J. Differential Geometry 9 (1974), 169–176. MR 49 #1534.

2. R. H. Gowdy, *Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: topologies and boundary conditions*, Ann. Phys. 83 (1974), 208–241.

3. A Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms. I. Derivations in the graded ring of differential forms*, Nederl. Akad. Wetensch. Proc. Ser. A 59 = Indag. Math. 18 (1956), 338–359. MR 18, 569.

4. P. D. Lax, *Hyperbolic systems of conservation laws*. II, Comm. Pure Appl. Math. 10 (1957), 537–566. MR 20 #176.

5. A. P. Stone, *Analytic conservation laws*, Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 2, 319–327. MR 35 #6160.

6. ———, *On the integrability of a vector 1-form*, J. London Math. Soc. (2) 6 (1973), 403–411. MR 47 #7650.

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