ABSTRACT. Recently the author [5] proved that \( \lambda \) connected continua \( X \) and \( Y \) are arc-like if and only if the topological product \( X \times Y \) is disk-like. Here we present an analogous theorem that generalizes the result of Fort [2] and Ganea [3] that disks are not torus-like. We prove that \( \lambda \) connected continua \( X \) and \( Y \) are circle-like if and only if \( X \times Y \) is torus-like.

We call a nondegenerate compact connected metric space a continuum. A map is a continuous single-valued function.

A continuum \( X \) is circle-like if for each positive number \( \epsilon \), there is an \( \epsilon \)-map (i.e., a map such that each point-preimage has diameter \(< \epsilon \)) of \( X \) onto a circle. Torus-like continua are defined in the same manner. Here a torus is the cartesian product of two circles.

A continuum is decomposable if it is the union of two proper subcontinua. A continuum is hereditarily decomposable if all of its subcontinua are decomposable. If each two points of a continuum \( X \) can be joined by a hereditarily decomposable subcontinuum of \( X \), then \( X \) is said to be \( \lambda \) connected.

A continuum \( Y \) is called a triod if it contains a subcontinuum \( Z \) such that \( Y - Z \) is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be atriodic.

A continuum is unicoherent provided that if it is the union of two subcontinua \( E \) and \( F \), then \( E \cap F \) is connected.

For any two metric spaces \((X, \psi)\) and \((Y, \phi)\), we shall always assume that the distance between two points \( p_1 = (x_1, y_1) \) and \( p_2 = (x_2, y_2) \) of the topological product \( X \times Y \) is defined by

\[
\rho(p_1, p_2) = ((\psi(x_1, x_2))^2 + (\phi(y_1, y_2))^2)^{1/2}.
\]

**Theorem 1.** Suppose that \( X \) and \( Y \) are \( \lambda \) connected continua and that \( X \times Y \) is torus-like. Then \( X \) is atriodic, every proper subcontinuum of \( X \) is unicoherent, and \( X \) is not unicoherent.

**Proof.** Let \( \psi \) and \( \phi \) be distance functions for \( X \) and \( Y \) respectively.

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Define \( Y_1 \) and \( Y_2 \) to be disjoint subcontinua of \( Y \). Note that if \( \epsilon = \phi(Y_1, Y_2) \) and \( f \) is an \( \epsilon \)-map of \( X \times Y \) onto a torus, then either \( f[X \times Y_1] \) or \( f[X \times Y_2] \) can be embedded in a 2-sphere [8, Lemma 1]. It follows from paragraphs 2 through 4 in the proof of Theorem 1 in [5] that \( X \) is atriodic. By the argument presented in paragraphs 5 through 13 in the same proof, every proper subcontinuum of \( X \) is unicoherent. Note that \( Y \) is atriodic and every proper subcontinuum of \( Y \) is unicoherent.

Now suppose that \( X \) is unicoherent. By Theorem 2 of [5], \( X \) is hereditarily decomposable. Hence there is a monotone map \( g \) of \( X \) onto the unit interval \([0, 1]\) [1, Theorem 8]. Define \( \epsilon_1 \) to be the minimum of

\[
\|\psi(g^{-1}[[0, n/9]], g^{-1}[[n + 1)/9, 1]])\| n = 1, 2, \ldots, 7.
\]

Assume that \( Y \) is unicoherent. Then \( Y \) is hereditarily decomposable and there exists a monotone map \( h \) of \( Y \) onto \([0, 1]\).

Define \( \epsilon \) to be a positive number less than \( \epsilon_1, \phi(h^{-1}(0), h^{-1}[[1/3, 1]]), \phi(h^{-1}(1), h^{-1}[[0, 2/3]]), \phi(h^{-1}(0, 1/3), h^{-1}[[2/3, 1]]) \). Let \( f \) be an \( \epsilon \)-map of \( X \times Y \) onto a torus \( T \).

At least one of the disjoint continua \( f[g^{-1}[[0, 4/9]] \times Y] \) and \( f[g^{-1}[[5/9, 1]] \times Y] \) is lying in a planar connected open subset of \( T \). We assume without loss of generality that a planar connected open set \( S \) in \( T \) contains \( f[g^{-1}[[0, 4/9]] \times Y] \).

The continuum \( K = f[g^{-1}[[2/9, 1/3]] \times Y] \) separates \( L = f[g^{-1}(0) \times Y] \) from \( M = f[g^{-1}(4/9) \times Y] \) in \( T \). Hence \( K \) separates \( L \) from \( M \) in \( S \). Note that the intersection of

\[
f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[0, 2/3]]] \quad \text{and} \quad f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[1/3, 1]]]
\]

is the continuum \( f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[1/3, 2/3]]] \). It follows from Janiszewski's theorem [7, Theorem 20, p. 173] that either

\[
E = f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[0, 2/3]]]
\]

or \( f[g^{-1}[[2/9, 1/3]] \times h^{-1}[[1/3, 1]]] \) separates \( L \) from \( M \) in \( S \).

We assume without loss of generality that \( E \) separates \( L \) from \( M \) in \( S \). But \( f[g^{-1}[[0, 4/9]] \times h^{-1}(1)] \) is a continuum in \( S \) that meets both \( L \) and \( M \) and misses \( E \), a contradiction. Hence \( Y \) is not unicoherent.

According to Lemma 2 of [6], \( Y \) is not separated by any of its subcontinua. By Theorem 5 of [4], there exists a monotone map \( k \) of \( Y \) onto a circle \( C \).

Define \( Z_1, Z_2, Z_3, \) and \( Z_4 \) to be arcs whose interiors are pairwise disjoint such that \( C = \bigcup_{i=1}^{4} Z_i \) and \( Z_1 \cap Z_3 = \emptyset = Z_2 \cap Z_4 \). Let \( V_1, V_2, V_3, \) and \( V_4 \) be arcs in \( C \) such that \( V_1 \cap V_2 = \emptyset = V_2 \cap V_4 \), and for each integer \( i (1 \leq i \leq 4) \), the interior of \( V_i \) contains \( Z_i \).
Define \( \epsilon' \) to be a positive number less than \( \epsilon_1, \phi(k^{-1}[V_1], k^{-1}[V_3]), \phi(k^{-1}[V_2], k^{-1}[V_4]), \) and the minimum of \( \{\phi(k^{-1}[Z_i], k^{-1}[C - V_i])\} \) \( i = 1, 2, 3, \) and \( 4 \). Let \( t \) be an \( \epsilon' \)-map of \( X \times Y \) onto the torus \( T \).

For each integer \( i (1 \leq i \leq 4) \) define \( A_i = t[X \times k^{-1}[V_i]] \). Note that \( T = \bigcup_{i=1}^{4} A_i \) and \( A_1 \cap A_3 = \emptyset = A_2 \cap A_4 \).

Using arcs in \( T \) that approximate each \( t[g^{-1}(0) \times k^{-1}[Z_i]] \), we define for each \( i (1 \leq i \leq 4) \) an arc \( \alpha_i \) in \( A_i \cap t[g^{-1}([0, 2/9]) \times Y] \) such that \( \alpha = \bigcup_{i=1}^{4} \alpha_i \) is a simple closed curve. By Fort's lemma \([2]\), there is a retraction \( r \) of \( T \) onto \( \alpha \). The torus \( T \) is not separated by \( \alpha \); for otherwise, \( r \) restricted to the closure of the planar component of \( T - \alpha \) would be a retraction of a disk onto its boundary, which is impossible. Note that \( \alpha \) lies in \( t[g^{-1}([0, 2/9]) \times Y] \).

In a similar manner, we define simple closed curves \( \beta \) and \( \gamma \) contained in \( B = t[g^{-1}([1/3, 2/3]) \times Y] \) and \( t[g^{-1}([7/9, 1]) \times Y] \), respectively, such that neither \( \beta \) nor \( \gamma \) separates \( T \).

Since \( \alpha, \beta, \) and \( \gamma \) are pairwise disjoint, \( T - (\alpha \cup \beta \cup \gamma) \) has three components. Let \( H \) be the component of \( T - (\alpha \cup \beta \cup \gamma) \) whose boundary is \( \alpha \cup \gamma \). Note that \( H \) does not meet \( \beta \).

Since \( B \) is a continuum in \( T \) that contains \( \beta \) and misses \( \alpha \cup \gamma \), \( B \) does not intersect \( H \). Thus \( H \) is contained in the union of disjoint continua

\[ A = t[g^{-1}([0, 1/3]) \times Y] \quad \text{and} \quad G = t[g^{-1}([2/3, 1]) \times Y]. \]

Since \( \alpha \) and \( \gamma \) lie in \( A \) and \( G \), respectively, it follows that \( H \) meets both \( A \) and \( G \). But this implies that \( H \) is not connected, a contradiction. Hence \( X \) is not unicoherent.

**Theorem 2.** Suppose that \( X \) and \( Y \) are \( \lambda \)-connected continua. Then \( X \) and \( Y \) are circle-like if and only if \( X \times Y \) is torus-like.

**Proof.** If \( X \times Y \) is torus-like, then \( X \) and \( Y \) are both atriodic nonunicoherent \( \lambda \)-connected continua with the property that every proper subcontinuum is unicoherent (Theorem 1). It follows from Theorem 2 of \([6]\) that \( X \) and \( Y \) are circle-like.

To see that the torus-like product condition is also necessary, note that if \( f \) is an \( \epsilon/2 \)-map of \( X \) onto a circle \( C \) and \( g \) is an \( \epsilon/2 \)-map of \( Y \) onto \( C \), then the function \( h \) of \( X \times Y \) onto the torus \( C \times C \) defined by \( h((x, y)) = (f(x), g(y)) \) is an \( \epsilon \)-map.

**Question.** Must continua \( X \) and \( Y \) (not necessarily \( \lambda \) connected) be circle-like when \( X \times Y \) is torus-like?

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**REFERENCES**


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