THE INTERSECTION OF THE SUBGROUPS OF FINITE INDEX IN SOME FINITELY PRESENTED GROUPS

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ABSTRACT. We consider the intersection of the subgroups of finite index in some finitely presented non-Hopfian groups.

The effort to establish the existence of a finitely generated non-Hopfian group seems to date back to the 1940's. R. Baer [5] was one of the first to consider this problem. Since then, several examples of finitely presented non-Hopfian groups have been given. If $G$ is any finitely generated group and if $\alpha$ is a surjective endomorphism of $G$, $\alpha$ induces in a natural way a surjective endomorphism $\bar{\alpha}$ of $G/N$, where $N$ is the intersection of the subgroups of finite index of $G$. But $G/N$ is residually finite and a finitely generated residually finite group is Hopfian. Hence $\bar{\alpha}$ is an automorphism and, consequently, the kernel of $\alpha$ is in $N$. Similarly if $K_i$ is the kernel of $\alpha^i$, $K_i \subseteq N$ so that if $K = \bigcup_{i \geq 1} K_i$, then $K \subseteq N$.

Professor W. Magnus has raised the question concerning under what circumstances $K = N$. The object of this note is to examine some of the known finitely presented non-Hopfian groups $G$ and to show that for these groups there exists an $\alpha$ such that $K$ is as large as possible, namely $K = N$.

Theorem. Consider the following groups $G$ with the respective endomorphisms $\alpha$:

(I) $\langle a, b; a^{-1}b^l a = b^m \rangle$, $l$, $m$ relatively prime, $a\alpha = a$, $b\alpha = b^l$;
(II) $\langle a, b, d; [a, b, b] = [a, b, a] = [a, d] = 1, a = [a^2, b, d] \rangle$, $a\alpha = a^2$, $b\alpha = b$, $d\alpha = d$;
(III) $\langle a, b, c; a^{-1}ca = b^{-1}cb = c^r \rangle$, $r \geq 2$, $a\alpha = a$, $b\alpha = b$, $c\alpha = c^r$.

Then in each case above, $K = N$. In (I), $K = N = G^\alpha$.

Proof. For the proofs that $\alpha$ gives a surjective endomorphism which is not an automorphism, see [9], [7] and [18] respectively. To show $K = N$, it suffices to show $N \subseteq K$. Let $g \in N$. We consider (I). If $B$ is the normal subgroup generated by $b$, $G/B$ is residually finite so that $g \in B$. Hence we can express $g$ as a word in the elements $x_i = a^l b a^{-i}$, $g = w(x_i)$. Let $r$ be the largest positive integer such that $x_r$ appears in $w$. Let $-s$ be the smallest negative integer such that $x_{-s}$ appears in $w$. Let $\theta$ be the inner automor-
phism: \( u \rightarrow a^{-r} ua^r \). Then by using the relation \( s_1^l = x_{i+1}^m \), one can easily check that \( g a_r^s \theta \) is a power of \( b \). But \( N \alpha = N, N \theta = N \) so \( g a_r^s \theta \) is a power of \( b \) in \( N \). However, in any finitely generated group \( G, N \subset G'' \) since \( G/G'' \) is a residually finite group. Hence \( g a_r^s \theta = b^l \in G'' \subset B' \). However, \( B/B' \) is isomorphic to the additive group of rationals generated by \( (l/m)^i \), \( i = 0, 1, 2, \ldots \), under the map \( x.B' \rightarrow (l/m)^i \) [4, p. 477] so that \( b^l \in B' \) implies \( j = 0 \). Hence \( g a_r^s \theta = 1 \) and, hence, \( g a_r^s = 1 \), so that \( K = N \). To show that \( N = G \) it suffices to show that \( G'' \) is contained in any normal subgroup \( \bar{G} \) of \( G \). But note that \( a^{-k} b^{l} a^{k} = b^{m^k} \) so that in any finite quotient group \( \bar{G} \) of \( G \), the order \( j \) of \( b \) is a divisor of \( m^k - l^k \) for some \( k \) so that \( (j, l) = 1 \). Hence in this quotient group we may write \( b = b^{s^l} \) so that \( a^{-k} b a = b^{m^k} \). This implies that \( \bar{G} \) has a cyclic commutator subgroup so that if \( G/L = \bar{G} \), then \( G'' \subset L \).

For the group (II) we define \( a_0 = a \) and \( a_{n+1} = (a_n, b, d) \). Now we assert the modulo \( K \), generators for \( A \), the normal subgroup generated by \( a \), are \( S = \{ a, (a, b) = c, n \geq 0 \} \). For one set of generators of \( A \) consists of the elements \( w^{-1} aw \), where \( w = w(b, d) \) is a freely reduced word in \( b \) and \( d \). We show by induction on the length of the word \( w \) that \( w^{-1} aw \) may be expressed as a product of elements in \( S \) and elements in \( K \). We first consider \( s = a_n \). Note \( a_n a^n = a \). Hence, \( (d, a_n a^n) = (d, a) = 1 \), so that \( d^{-1} a_n d = a_n \mod K \) and \( d a_n d^{-1} = a_n \mod K \). Also \( b a_n b^{-1} = (b^{-1} a_n^{-1} a) \). However, \( [(b^{-1} a_n^{-1} a) c_n] a_n^{-1} = 1 \), so that \( (b^{-1} a_n^{-1} c_n^{-1} = 1 \mod K, \) so that \( b a_n b^{-1} = c_n^{-1} a_n \mod K \). Similarly, \( b^{-1} a_n b = c_n a_n \mod K \). Now to consider \( s = c_n \), note

\[
d^{-1} c_n d = (d, c_n^{-1}) c_n
\]

But we claim

\[
(d, c_n^{-1}) = (d, c_n^{-1}) \mod K.
\]

To see this, note that if \( \overline{a} = b^2 d^{-1} (a, b) db^{-2} \), then a brief calculation shows \( \overline{a}^2 = a \) [7, p. 197] so that \( \overline{a} a = \overline{a}^2 = a. \)
Now, note that \( c_0^{-1}d_0 = d\bar{a} \mod K \) and \((d, \bar{a}) = 1 \mod K\). This implies that \((d, c_0^{-1}) = (d, c_0^{-1}) \mod K\). Now if we apply \( \alpha^n \) to \((d, c_n) \) and \((d, c_n^{-1}) \), respectively, we may deduce (2), so that from (1) and (2) we see \( d_n^{-1}c_n = a_n+1c_n \mod K \). Similarly \( d_n^{-1}d_n^{-1} = a_n+1^{-1}c_n \mod K \). Clearly, \( b_n^{-1}c_n = c_n \mod K \) so that our assertion about \( S \) is valid.

Now note that \( G/A \) is a free group of rank two which is consequently residually finite. This implies that \( N \subset A \). Hence if \( w \in N \), we may write \( w = s_1^{i_1}s_2^{i_2}\cdots s_k^{i_k} \), where \( s_i \in S \) and \( k \in K \). Note that \( a_n\alpha^n = a_n \alpha^n = (a, b), n \geq 1 \), so that we may find a positive integer \( r \) such that \( wa^r = a^r(a, b)^m \in N \). Hence in every finite quotient \( \overline{G} \) of \( G \) we have \( a^r = (a, b)^m \). Hence in \( \overline{G} \), \((a^r, b) = (a, b)^r = 1 \). Hence in \( \overline{G} \), \( a^r = (c_{2j}^2, d) = 1 \) and \( a^m = (c_{2m}^2, d) = 1 \). Hence in \( G \), \( a^r \in N \) and \( a^m \in N \). We will show this implies \( j = m = 0 \) so that \( wa^r = 0 \). To do this, for odd \( n \) let \( L_n \) be the normal subgroup of \( G \) generated by \( a^{n^2}, b^{n^2}, d^{n^2}, n > 1 \). Let

\[
P_n = (a, b; (a, b, a) = (a, b, b) = a^{n^2} = b^{n^2} = 1),
\]

\[
M_n = (c, d; (c, d, c) = (c, d, d) = c^{n^2} = d^{n^2} = 1).
\]

Let \( H_n \) be the subgroup of \( P_n \) generated by \((a^2, b)\) and \( a \). Let \( J_n \) be the subgroup of \( M_n \) generated by \((c, d)\) and \( c \).

Form the free product of \( P_n \) and \( M_n \) amalgamating \( H_n \) and \( J_n \) as \( c = (a^2, b) \) and \( a = (c, d) \). This free product with \( H_n = J_n \) is just \( G/L_n \). Now a free product of finite groups with a subgroup amalgamated is residually finite. If \( n > 1 \) is odd, \( a^n \neq 1 \) in \( G/L_n \) so that we can find a finite quotient of \( G/L_n \) (and hence of \( G \)) in which \( a^n \neq 1 \). Hence, if \( t \) is odd, \( a^t \notin N \) in \( G \).

Moreover, if \( t \) is even, \( t \neq 0 \), \( t = 2^{qj}, j \) odd, then \( a^t \notin N \), for as can be checked from the defining relations of \( G \), the order of \( a \) in any finite quotient of \( G \) is odd so that \( a^t \in N \) would imply \( a^t \notin N \).

For the group (III), if \( C \) is the normal subgroup generated by \( c \), we see that \( G/C \) is a free group of rank two so that \( N \subset C \). Moreover, generators for \( C \) are \( w^{-1}cw \), where \( w = w(a, b) \) is a freely reduced word in \( a \) and \( b \).

Now we claim that modulo \( K \) generators of \( C \) are the elements \( c_j = a^i\alpha^{-j} \), \( j \geq 0 \). To show this we show by induction on the length of the word \( w \) that any element \( w^{-1}cw \) is expressible in terms of the \( c_j \) and elements of \( K \). If the length of \( w \) is zero, that is, if \( w \) is the empty word, this is clear. To complete the induction it suffices, as in the discussion for the group (II), to show that \( x^{-1}c_jx \) is expressible in terms of the \( c_j \) and elements in \( K_j \) where \( x \) is one of the elements \( a, b, a^{-1}, b^{-1} \). To this end note that \( ac_ja^{-1} = c_{j+1} \) and \( a^{-1}c_ja = c_{j-1} \) if \( j \geq 1 \) while \( a^{-1}c_0a = c_0 \). Also we note that if \( b_j = b^i\alpha^{-j} \), \( j \geq 0 \), then \( b_j = a_j \mod K \). For note, \( b_0a = b^i\alpha^{-j}b^{-j} = b^ib^{-1}cbb^{-j} = b_{j-1} \), \( j \geq 1 \). Hence \( b_0a = b_0 = c \). Similarly \( c_ja = c \) so that \( (b_j^{-1})a^{-1} = \)}
1. Hence modulo $K$ we may write $bc_j b_j^{-1} = bb_j b_j^{-1} = b_j b_{j+1} = c_j b$ and 
$b_{j-1} b = b_j b_{j+1}$. The latter is $b_{j-1} = c_{j-1}$ if $j \geq 1$ and is $b_0 = c_0$ if $j = 0$. 
This completes the proof that elements of $N$ are expressible in terms of the 
c_j and elements in $K$.

In view of the fact that $c_j a^j = c$, it follows that if $w \in N$, then $w a^t = c^u$ for some positive integer $t$. Hence $c^u \in N$. However, if we consider the 
groups 

$$P_n = \langle a, c; a^{-1} c a = c^t, a^n = 1 \rangle, \quad n \geq 2,$$

$$M_n = \langle b, d; b^{-1} d b = d^t, b^n = 1 \rangle, \quad n \geq 2,$$

then $c$ and $d$ have the same orders and $c^n \neq 1$. Since $P_n$ and $M_n$ are finite 
groups, their free product with the amalgamation $c = d$ is residually finite. 
However this free product yields a homomorphic image of $G$. Hence if $c^u \in 
N$ in $G$, we must have $u = 0$.

Problem. Does every finitely presented non-Hopfian group have a sur-
jective endomorphism $\alpha$ for which $K = N$?

Some of the references listed below are not referred to in the present 
paper but are relevant to the topic of Hopficity. References for any asser-
tions made about residually finite groups can be found in the survey paper 
[30].

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