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IDENTITIES IN COMBINATORICS. II: A q-ANALOG OF THE LAGRANGE INVERSION THEOREM

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ABSTRACT. A q-analog of Lagrange's inversion theorem is obtained. It is applied to give a new proof of an expansion theorem due to Carlitz and to obtain formulae for certain combinatorial numbers studied by Carlitz.

1. **Introduction.** In [3], L. Carlitz addressed the question of possible q-analogs for the celebrated Lagrange inversion formula [1, p. 158]:

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!(\phi(x))^n} \left[\frac{d^{n-1}}{dt^{n-1}} (f'(t)(\phi(t))^n)\right]_{t=0}, \tag{1.1}$$

a formula valid provided $f(x)$ and $\phi(x)$ are analytic around $x = 0$ and $\phi(0) \neq 0$.

Carlitz [3, p. 206, equation (1.11)] was able to show that

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{(q)_n (x)_n} [\Delta^{n-1}(\Delta f(t) \cdot (t)_n)]_{t=0}, \tag{1.2}$$

where $(x)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1})$ and $\Delta f(t) = t^{-1}(f(t) - f(tq))$. This result is a q-analog of (1.1) in the case $\phi(x) = 1 - x$.

In [4], Carlitz asks for a general q-analog of Lagrange inversion. Namely he asks for a formula for the C_n in

$$F(x) = \sum_{n \geq 0} \frac{C_n x^n}{(q)_n \phi(xq)\phi(xq^2) \cdots \phi(xq^n)}, \qquad C_0 = F(0).$$

In particular he asks for an operational formula for computing the $C(n|q)$ in

$$x = \sum_{n=1}^{\infty} x^n (xq)_n C(n-1|q), \tag{1.3}$$

and he points out that

$$C(n-1 \mid q) \neq \frac{(1-q)}{(q)_n} \{\Delta^{n-1}(xq)_n^{-1}\} = q^{n-1} \frac{(1-q)}{(1-q^n)} \binom{2n-2}{n-1}_q,$$

where $\binom{N}{M}_q = (q)_N (q)_M^{-1} (q)_{N-M}^{-1}$.

In [2], Carlitz discusses $C(n|q)$ extensively, which is initially defined by

$$C(n \mid q) = \sum_{(a_i)} q^{a_1 + a_2 + \cdots + a_n}$$

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summed over all $(a_1, a_2, \cdots a_n)$ with $1 \le a_1 \le a_2 \le \cdots \le a_n$ and $a_i \le i$ for each i.

When $q = 1$, then, of course, the Lagrange inversion formula (1.1) is applicable to (1.3) and indeed $C(n|1) = (n+1)^{-1}\binom{2n}{n}$, the nth Catalan number.

In this paper we shall (in §2) prove the following q-analog of the Lagrange inversion formula (1.1); our result reduces to (1.2) in the case $\phi(x) = 1 - xq^{-1}$ and it reduces to (1.1) when $q \to 1$. Before stating our main result we require the following notation:

$$\Delta_{(n)} f(t) = (f(t) - f(tq^n))/t.$$

Note that $\Delta = \Delta_{(1)}$; indeed $\Delta_{(n)}$ is just q-differentiation to the base q^n. Furthermore $\Delta_{(0)} f(x) \equiv 0$.

Theorem 1. *Let $F(x)$ and $\phi(x)$ be analytic around $x = 0$ with $\phi(0) \neq 0$, and let k be a nonnegative integer. Define*

$$\delta_r = (q)_{r-1} \cdot \{\textit{Coefficient of } t^{r-1} \textit{ in } (\Delta F(t)) \cdot \phi(tq^k)\phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})\}$$

$$= [\Delta^{r-1}((\Delta F(t))\phi(tq^k)\phi(tq^{k+1}) \cdots \phi(tq^{k+r-1}))]_{t=0},$$

and for $j \le r$,

$$\epsilon_j(r) = -(q)_{r-1} q^{r-j}$$

$$\cdot \left\{ \textit{Coefficient of } t^{j-2} \textit{ in } (\Delta_{(r-j)} t^{-1}\phi(tq)) \frac{\phi(tq^k)\phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})}{\phi(tq)\phi(tq^2) \cdots \phi(tq^{r-j+1})(q)_{r-j}} \right\}$$

$$= -(q^{j-1})_{r-j+1} q^{r-j}$$

$$\cdot \left[\Delta^{j-2} \left\{ (\Delta_{(r-j)} t^{-1}\phi(tq)) \frac{\phi(tq^k)\phi(tq^{k+1}) \cdots \phi(tq^{k+r-1})}{\phi(tq)\phi(tq^2) \cdots \phi(tq^{r-j+1})(q)_{r-j}} \right\} \right]_{t=0}.$$

Then, for $|q| < 1$,

$$F(x) = F(0) + \sum_{n=1}^{\infty} \frac{x^n C_n}{(q)_n \phi(xq)\phi(xq^2) \cdots \phi(xq^n)}, \tag{1.4}$$

where

$$C_n = \begin{vmatrix} 1 & 0 & 0 & & 0 & \delta_0 \\ \epsilon_1(1) & 1 & 0 & & 0 & \delta_1 \\ \epsilon_2(2) & \epsilon_1(2) & 1 & & 0 & \delta_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & 1 & \\ \epsilon_n(n) & \epsilon_{n-1}(n) & \epsilon_{n-2}(n) & \cdots & \epsilon_1(n) & \delta_n \end{vmatrix},$$

When $q \to 1$, we shall show that $\epsilon_j(n)(q)_j^{-1} \to 0$ whenever $j < n$; hence it follows that

$$C_n(q)_n^{-1} \to n!^{-1}\,[(d^{n-1}/dt^{n-1})(f'(t)(\phi(t))^n)]_{t=0}$$

as $q \to 1$ (see (2.4) at the end of §2). Thus Theorem 1 is the q-analog of Lagrange inversion.

In §3, we shall apply Theorem 1 to prove (1.2) and certain formulae related to Carlitz's work in [3]. In §4, we shall apply Theorem 1 to the $C(n|q)$.

2. **Proof of Theorem 1.** The elementary proof of (1.1) found in [1, §55.1] will be adapted to our needs here. We begin with the observation that the C_n in (1.4) must exist since $|x^n/\phi(xq)\cdots\phi(xq^n)| < \rho^n < 1$ for x in some sufficiently small neighborhood of 0. Replacing x by t in (1.4) and applying Λ we see that

$$\Delta F(t) = \sum_{n\geq 0} \frac{C_n t^{n-1}}{(q)_n}\left\{\frac{1}{\phi(tq)\phi(tq^2)\cdots\phi(tq^n)} - \frac{q^n}{\phi(tq^2)\phi(tq^3)\cdots\phi(tq^{n+1})}\right\}$$

$$= \sum_{n\geq 0} \frac{C_n t^n q^n\{t^{-1}q^{-n}\phi(tq^{n+1}) - t^{-1}\phi(tq)\}}{(q)_n\phi(tq)\phi(tq^2)\cdots\phi(tq^{n+1})}$$

$$= -\sum_{n\geq 0} \frac{C_n t^{n+1} q^n \Delta_{(n)}[t^{-1}\phi(tq)]}{(q)_n\phi(tq)\phi(tq^2)\cdots\phi(tq^{n+1})}.$$

Hence

$$(\Delta F(t))\phi(tq^k)\phi(tq^{k+1})\cdots\phi(tq^{k+r-1})t^{-r}(q)_{r-1}$$

$$= -(q)_{r-1}\sum_{n\geq 0} \frac{C_n t^{n-r+1} q^n(\Delta_{(n)}t^{-1}\phi(tq))\phi(tq^k)\phi(tq^{k+1})\cdots\phi(tq^{k+r-1})}{(q)_n\phi(tq)\phi(tq^2)\cdots\phi(tq^{n+1})}. \tag{2.1}$$

We now ask for the coefficient of t^{-1} on each side of the above identity. On the left-hand side we see that δ_r is the coefficient of t^{-1}. On the right-hand side we see that for $n > r$ each term is analytic in t and so the coefficient of t^{-1} is zero; hence we need only examine those terms with $n \leq r$. When $n = r$, the term is

$$\frac{C_r t^{-1}\{\phi(tq^{r+1}) - q^r\phi(tq)\}\phi(tq^k)\cdots\phi(tq^{k+r-1})}{(1-q^r)\phi(tq)\phi(tq^2)\cdots\phi(tq^{r+1})} = \frac{C_r t^{-1}}{1-q^r}(1 - q^r + \alpha t + \beta t^2 + \cdots)$$

and so the coefficient here is C_r.

When $n < r$, we desire the coefficient of t^{r-n-2} in

$$-(q)_{r-1}C_n q^n(\Delta_{(n)}t^{-1}\phi(tq))\,\frac{\phi(tq^k)\phi(tq^{k+1})\cdots\phi(tq^{k+r-1})}{\phi(tq)\phi(tq^2)\cdots\phi(tq^{n+1})(q)_n}$$

which is just $C_n\epsilon_{r-n}(r)$. Hence

$$\delta_r = \sum_{n=1}^{r} C_{r-n}\epsilon_n(r) + C_r, \tag{2.2}$$

and the determinental expression for C_N is now obtained by applying Cramer's rule to the above system of equations with $0 \leq r \leq N$. This concludes the proof of Theorem 1.

Let us now show that $\lim_{q\to 1}\epsilon_j(r)(q)_j^{-1} = 0$ for $0 < j \leq r$.

$$\begin{aligned}
\lim_{q\to 1} \epsilon_j(r)(q)_j^{-1} &= -\text{Coefficient of } t^{j-2} \text{ in } \binom{r-1}{j}\cdot\left[\frac{d}{dt}(t^{-1}\phi(t))\right]\phi(t)^{j-1} \\
&= -\text{Coefficient of } t^{j-2} \text{ in } \frac{1}{j}\binom{r-1}{j}t^{j-1}\frac{d}{dt}(t^{-1}\phi(t))^j \qquad (2.3)\\
&= -\text{Coefficient of } t^{-1} \text{ in } \frac{1}{j}\binom{r-1}{j}\frac{d}{dt}(t^{-1}\phi(t))^j = 0,
\end{aligned}$$

since $dt^r/dt \neq Ct^{-1}$ for any r with $C \neq 0$ and since $(t^{-1}\phi(t))^j = \Sigma_{r=-j}^{\infty}\alpha_r t^r$.

(2.3) may now be applied to show that

$$\lim_{q\to 1}\frac{C_n}{(q)_n} = \lim_{q\to 1}\frac{\delta_n}{(q)_n} = n!^{-1}\left[\frac{d^{n-1}}{dt^{n-1}}(f'(t)(\phi(t))^n)\right]_{t=0}. \tag{2.4}$$

Since $\epsilon_r(r) \equiv 0$ (due to the fact that $\Delta_{(0)}f(t) \equiv 0$), we see that $\delta_1 = c_1$ and so (2.4) is clearly true for $n = 1$. If we assume (2.4) to be valid up to, but not including a particular n, then, by (2.2),

$$\frac{\delta_n}{(q)_n} = \sum_{j=1}^{n-1}\frac{C_{n-j}}{(q^{j+1})_{n-j}}\frac{\epsilon_j(n)}{(q)_j} + \frac{C_n}{(q)_n}.$$

Consequently, by the induction hypothesis,

$$\begin{aligned}
\lim_{q\to 1}\frac{C_n}{(q)_n} &= \lim_{q\to 1}\frac{\delta_n}{(q)_n} - \sum_{j=1}^{n-1}\frac{(n-j)!j!}{n!}\left(\lim_{q\to 1}\frac{C_{n-j}}{(q)_{n-j}}\right)\left(\lim_{q\to 1}\frac{\epsilon_j(n)}{(q)_j}\right) \\
&= \lim_{q\to 1}\frac{\delta_n}{(q)_n} - \sum_{j=1}^{n-1}\frac{j!}{n!}\left[\frac{d^{n-j-1}}{dt^{n-j-1}}(f'(t)(\phi(t))^{n-j})\right]_{t=0}\cdot 0 \\
&= \lim_{q\to 1}\frac{\delta_n}{(q)_n} = n!^{-1}\left[\frac{d^{n-1}}{dt^{n-1}}(f'(t)(\phi(t))^n)\right]_{t=0}.
\end{aligned}$$

3. **The Carlitz expansion (1.2).** We now apply Theorem 1 to the special case $\phi(x) = 1 - ax$, $k = 1$. In this case, when $j < r$,

$$\epsilon_j(r) = \frac{-(q)_{r-1}q^{r-j}}{(q)_{r-j}}\{\text{Coefficient of } t^{j-2} \text{ in } t^{-1}(t^{-1}(1-atq) - t^{-1}q^{-r+j}(1-atq^{r-j+1}))(atq^{r-j+2})_{j-1}\}$$
$$= \frac{-(q)_{r-1}q^{r-j}}{(q)_{r-j}}\{\text{Coefficient of } t^{j-2} \text{ in } t^{-2}(1-q^{-r+j})(atq^{r-j+2})_{j-1}\}$$
$$= 0,$$

since the highest power of t in $t^{-2}(1-q^{-r+j})(atq^{r-j+2})_{j-1}$ is t^{j-3}. Hence, by Theorem 1, $C_n = \delta_n$, and we have

$$F(x) = F(0) + \sum_{n\geq 1} \frac{x^n}{(q)_n(axq)_n}[\Delta^{n-1}(\Delta F(t)\cdot(atq)_n)]_{t=0}. \tag{3.1}$$

When $a = q^{-1}$ we get (1.2).

4. **Evaluation of the $C(n|q)$.** We now treat the case originally envisioned by Carlitz; namely (1.3) where now $F(x) = x$, $\phi(x) = (1-x)^{-1}$.

Theorem 2. *Let $C(n|q)$ be defined by* (1.2), *then*

$$q^{2n-2}\frac{(1-q)}{1-q^n}\binom{2n-2}{n-1}_q = C(n-1\,|\,q) - q\sum_{j=1}^{n}(1-q^{n-j})q^{(n-j+1)(j-1)}\binom{2j-1}{j-1}_q C(n-1-j\,|\,q). \tag{4.1}$$

Remark. This result exhibits clearly that $C(n-1|1) = n^{-1}\binom{2n-2}{n-1}$.

Proof. We assert that (4.1) is essentially (2.2) when $\phi(x) = (1-x)^{-1}$, $k = 2$. This is because in this case

$$\delta_r = [\Delta^{r-1}((\Delta t)\cdot(tq^2)_r^{-1})]_{t=0} = q^{2r-2}(1-q)(1-q^{2r-2})\cdots(1-q^r),$$

and

$$\epsilon_j(r) = -(q)_{r-1}q^{r-j}\cdot\left\{\text{Coefficient of } t^{j-2} \text{ in } \frac{[1-(1-tq)/q^{r-j}(1-tq^{r-j+1})]}{t^2(tq^{r-j+2})_j}\right\}(q)_{r-j}^{-1}$$
$$= (q)_{r-1}\left\{\text{Coefficient of } t^{j-2} \text{ in } \frac{(1-q^{r-j})(1-tq-tq^{r-j+1})}{t^2(tq^{r-j+1})_{j+1}}\right\}(q)_{r-j}^{-1}$$
$$= (q)_{r-1}\left\{\text{Coefficient of } t^j \text{ in } (1-q^{r-j})(1-tq(1+q^{r-j}))\sum_{n\geq 0}\binom{j+n}{n}_q t^n q^{(r-j+1)n}\right\}(q)_{r-j}^{-1}$$
$$= (q)_{r-1}\left\{(1-q^{r-j})\binom{2j}{j}_q q^{(r-j+1)j} - q(1-q^{2(r-j)})\binom{2j-1}{j-1}_q q^{(r-j+1)(j-1)}\right\}(q)_{r-j}^{-1}.$$

Hence, dividing (2.2) by $(q)_r$ and noting that $C_n = C(N-1|q)(q)_n$, we obtain

$$q^{2r-2}\frac{(1-q)}{1-q^r}\binom{2r-2}{r-1}_q = C(r-1\,|\,q)$$

$$+\sum_{j=1}^{r}\frac{(1-q^{r-j})}{1-q^r}\left\{\binom{2j}{j}_q q^{(r-j+1)j} - q(1+q^{r-j})\binom{2j-1}{j-1}_q q^{(r-j+1)(j-1)}\right\}C(r-1-j\,|\,q)$$

$$= C(r-1|n) - q\sum_{j=1}^{r}(1-q^{r-j})\,q^{(r-j+1)(j-1)}\binom{2j-1}{j-1}_q C(r-1-j|q),$$

as desired.

We can, of course, obtain a determinental expression for $C(n-1|q)$ as given by Theorem 1; however Theorem 2 seems to most clearly exhibit the relationship between $C(n-1|q)$ and the q-Catalan number $\binom{n}{1}_q^{-1}\binom{2n-2}{n-1}_q$.

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