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POWER-ASSOCIATIVITY OF ANTIFLEXIBLE RINGS

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ABSTRACT. Conditions which force an antiflexible ring of characteristic p to be power-associative are determined.

1. **Introduction.** We prove that antiflexible rings are always mth power-associative except when $m = p^r$ for $r \in Z^+$, where the prime $p > 2$ is the characteristic of R. It is proved that R is power-associative provided there exist some $i, j \in \{1, 2, \ldots, p^r - 1\}$ such that $(|i - j|, p) = 1$ and $x^{p^r-i}x^i = x^{p^r-j}x^j$ for $r \in R$ and $r \in Z^+$. We show that this condition may not be omitted by constructing a family of antiflexible rings such that $x^{p^r-1}x = 0$ but $xx^{p^r-1} \neq 0$. In particular, it is possible to construct antiflexible rings with elements which are right nilpotent but not left nilpotent.

2. **Preliminaries.** A ring is *antiflexible* provided

$$(x, y, z) = (z, y, x) \tag{2.1}$$

for all x, y, z in the ring where the associator (a, b, c) is defined by $(a, b, c) = (ab)c - a(bc)$. Let x be an element of a ring. We define x^n for all positive integers n by

$$x^1 = x; \quad x^k = x^{k-1}x, \qquad k = 2, 3, 4, \ldots. \tag{2.2}$$

A ring is mth *power-associative* provided

$$(x^i, x^j, x^k) = 0 \tag{2.3}$$

for all x in the ring and for all positive integers i, j, k such that $i + j + k \leq m$, m a positive integer. This is equivalent to saying that $x^{i+j} = x^i x^j$ for all i, j such that $i + j \leq m$. A ring is *power-associative* provided it is mth power-associative for every positive integer m. The following two identities hold in any ring:

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z, \tag{2.4}$$

$$[xy, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y) \tag{2.5}$$

where the *commutator* $[a, b]$ is defined by $[a, b] = ab - ba$.

Let k be a positive integer. If a ring is of *characteristic* k, then $kx = 0$ for all x in the ring, and if n is prime to k, then $nx = 0$ implies $x = 0$ for x in the ring; and if a ring is of *characteristic zero*, then $nx = 0$ implies $x = 0$

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for every positive integer n and for x in the ring.

From here on, R will denote an antiflexible ring of characteristic prime to 2 which is also third power-associative. Hence

$$(x, x, x) = 0 \tag{2.6}$$

for all $x \in R$. Linearization of (2.6) yields

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \tag{2.7}$$

upon application of (2.1) and the fact that R is of characteristic prime to 2. Subtracting (2.7) from (2.5) and using (2.1), we obtain

$$[xy, z] - x[y, z] - [x, z]y = -2(x, z, y). \tag{2.8}$$

Set $y = x$ or $z = x^2$ in (2.8). Then $(x, x^2, x) = 0$ and hence

$$(x, x^2, x) = 0 = (x^2, x, x) \tag{2.9}$$

from (2.7) with $y = x$ and $z = x^2$ and (2.1). Therefore, R is 4th power-associative as was first established by Kosier in [3]. Thus, as Kosier observed, it follows from a theorem of Albert [1] that if R is of characteristic 0 then R is power-associative. On the other hand, Rodabaugh published an example [5] showing that if R is not of characteristic 0 then R need not be power-associative. Theorems of power-associativity of arbitrary rings and algebras have appeared in Albert [1], Kokoris [2], Leadley and Richie [4].

3. **Main section.** We make use of the following result of Albert [1].

Lemma 1. *Let the characteristic of a ring A be prime to two, $n \geq 4$, $x^\lambda x^\mu = x^{\lambda+\mu}$ for $\lambda + \mu < n$. Then*

$$n[x^{n-1}, x] = 0, \quad [x^{n-\alpha}, x^\alpha] = \alpha[x^{n-1}, x], \qquad \alpha \in \{1, \dots, n-1\},$$

so that, if n is prime to the characteristic of A, we have $x^{n-\alpha}x^\alpha = x^\alpha x^{n-\alpha}$ for $\alpha \in \{1, 2, \dots, n-1\}$.

Recall that by R we always mean an antiflexible ring of characteristic prime to 2 which is also third power-associative. By $R[x]$ we mean the subring of R generated by $x \in R$.

Lemma 2. *If $[x^{m-1}, x] = 0$, then $R[x]$ is mth power-associative provided $R[x]$ is $(m-1)$st power-associative.*

Proof. Using (2.8),

$$0 = [x^{m-1}, x] = [x^{m-1-i}x^i, x] = -2(x^{m-1-i}, x, x^i)$$
$$= -2x^{m-i}x^i + 2x^{m-1-i}x^{i+1}.$$

Hence,

$$x^{m-i}x^i = x^{m-1-i}x^{i+1}, \qquad 1 \leq i \leq m-2, \tag{3.1}$$

which implies that $x^{m-k}x^k = x^m$ for all $1 \leq k \leq m-1$ and hence $R[x]$ is mth power-associative.

Lemma 3. *For and* $m \in Z^+$, *if* $R[x]$ *is* $(m-1)$*st power-associative then*

(3.2) $$(i+1)x^m = 2x^{m-i}x^i + (i-1)xx^{m-1}, \qquad 1 \le i \le m-1.$$

Proof. We establish the identity by induction. $i = 1$ case is true by definition. Suppose that the identity holds for $i = k-1$. Then

$$x^{m-k+1}x^{k-1} = (x^{m-k}, x, x^{k-1}) + x^{m-k}x^k = (xx^{m-k-1}, x, x^{k-1}) + x^{m-k}x^k$$

since $R[x]$ is $(m-1)$st power-associative. By (2.4), we obtain, using (2.1) and (2.7),

$$\begin{aligned}(xx^{m-k-1}, x, x^{k-1}) &= (x, x^{m-k}, x^{k-1}) - (x, x^{m-k-1}, x^k)\\ &= -(x^{m-k}, x, x^{k-1}) - (x^{m-k}, x^{k-1}, x) - (x, x^{m-k-1}, x^k)\\ &= -x^{m-k+1}x^{k-1} + x^{m-k}x^k - x^{m-1}x + x^{m-k}x^k - x^{m-k}x^k + xx^{m-1}.\end{aligned}$$

Thus,

$$x^{m-k+1}x^{k-1} = -x^{m-k+1}x^{k-1} + 2x^{m-k}x^k - x^m + xx^{m-1}$$

or

(3.3) $$2x^{m-k+1}x^{k-1} = 2x^{m-k}x^k - x^m + xx^{m-1}.$$

By the induction hypothesis,

$$kx^m = 2x^{m-k+1}x^{k-1} + (k-2)xx^{m-1} = 2x^{m-k}x^k - x^m + (k-1)xx^{m-1}$$

using (3.3), hence

$$(k+1)x^m = 2x^{m-k}x^k + (k-1)xx^{m-1}.$$

Theorem 1. *Let the characteristic of* R *be a prime* p. *Then* R *is pth power-associative if and only if for all* $x \in R$ *there exists some* $i, j \in \{1, 2, \ldots, p-1\}$, $i \ne j$ *such that* $x^{p-i}x^i = x^{p-j}x^j$.

Proof. One direction is obvious. Assume that $x^{p-i}x^i = x^{p-j}x^j$ for some $i, j \in \{1, 2, \ldots, p-1\}$, $i \ne j$. By Lemma 1, R is $(p-1)$st power-associative. Then we may use (3.2):

$$(i+1)x^p = 2x^{p-i}x^i + (i-1)xx^{p-1}, \quad (j+1)x^p = 2x^{p-j}x^j + (j-1)xx^{p-1}.$$

Subtraction yields $(i-j)x^p = (i-j)xx^{p-1}$ since $x^{p-i}x^i = x^{p-j}x^j$. However, $|i-j| \in \{1, 2, \ldots, p-1\}$, hence $x^p = xx^{p-1}$. Thus R is pth power-associative by Lemma 2.

Theorem 2. *Let the characteristic of* R *be a prime* $p > 2$. *Then* R *is power-associative if and only if for every* $x \in R$, $x^{p^r-i}x^i = x^{p^r-j}x^j$ *for all* $r \in Z^+$ *and for some* $i, j \in \{1, 2, \ldots, p^r-1\}$ *with* $(|i-j|, p) = 1$.

Proof. The case $r = 1$ yields that R is pth power-associative. Since any integer m between tp and $(t+1)p$ for $t \in Z^+$ is relatively prime to p, Lemma 1 implies that R is mth power-associative providing R is tpth power-associative.

Next, we establish that R is lpth power-associative for $l \in \{1, 2, \ldots, p-1\}$. Using induction on l, we may assume that R is $(l-1)p$th power-associative. Hence R is also $(lp-1)$th power-associative. So, for all $x \in R$, we have

$$x^{l(p-1)} = x^{lp-l} = (x^l)^{p-1}.$$

Therefore,

$$x^{lp-l}x^l = x^{l(p-1)}x^l = (x^l)^{p-1}x^l = x^l(x^l)^{p-1} = x^l x^{lp-l};$$

hence $[x^{lp-l}, x^l] = 0$. Now Lemma 1 yields the claim.

We may now assume that R is $(p^r - 1)$th power-associative and prove that it is p^rth power-associative. Let $i, j \in \{1, \ldots, p^r - 1\}$ with $i \neq j$ and $(|i-j|, p) = 1$, and $x^{p^r - i}x^i = x^{p^r - j}x^j$. Using (3.2),

$$(i+1)x^{p^r} = 2x^{p^r - i}x^i + (i-1)xx^{p^r - 1},$$

$$(j+1)x^{p^r} = 2x^{p^r - j}x^j + (j-1)xx^{p^r - 1}$$

which yield $x^{p^r} = xx^{p^r - 1}$. This completes the proof.

We close with an example to show that Theorem 2 is the best possible for antiflexible rings. Rodabaugh's example [5] shows that there exist third power-associative, antiflexible rings of characteristic p, p an odd prime, which are not pth power-associative. Our example goes one step further. It shows that if R is of characteristic p, p a prime, then it is possible for $x^{p-i}x^i = 0$ for some i while $x^{p-j}x^j \neq 0$ for all $j \neq i$. Furthermore, the example provides antiflexible algebras of dimension p^r, for $r \in Z^+$, which are not p^rth power-associative and which contain elements which are right nilpotent but not left nilpotent.

Example. Let F be a field of characteristic $p > 3$, p a prime. Let $B_p(\alpha)$ be the p^rth dimensional algebra over F with basis $1, x, x^2, \ldots, x^{p^r - 1}$, where $r \in Z^+$ and

(1) $$x^k x^l = x^{k+l} \quad \text{if } 2 \leq k + l \leq p^r - 1,$$

(2) $$x^k x^l = \tfrac{1}{2}(1 - l)\alpha, \qquad \alpha \in F, \ \alpha \neq 0, \quad \text{if } k + l = p^r,$$

(3) $$x^k x^l = \tfrac{1}{2}\alpha x^{k+l-p^r} \quad \text{if } k + l > p^r, \qquad k, l \in \{1, 2, \ldots, p^r - 1\}.$$

Observe that $x^{p^r} = 0$, and $xx^{p^r - 1} = \alpha \neq 0$.

To show that $B_p(\alpha)$ is a third power-associative antiflexible algebra one should verify that

(3.4) $(x^k, x^l, x^m) = (x^m, x^l, x^k)$ for all $k, l, m \in \{1, 2, \ldots, p^r - 1\}$.

It is immediate that $(x^k, x^k, x^k) = 0$ for all $k \in \{1, 2, \ldots, p^r - 1\}$ since $p > 3$. Identity (3.4) includes combinations of the following cases:

(i) $k + l + m = p^r$;

(ii) $k + l = p^r$, $l + m = p^r$, $k + m = p^r$;

(iii) $k + l > p^r$, $l + m > p^r$, $k + m > p^r$, $k + l + m = 2p^r$, $k + l + m < 2p^r$, $k + l + m > 2p^r$,

We illustrate one typical case:

$$k + l > p^r \quad \text{and} \quad k + l + m = 2p^r, \qquad x^k x^l = \tfrac{1}{2}\alpha x^{k+l-p^r},$$

$$(x^k x^l)x^m = \tfrac{1}{2}\alpha x^{k+l-p^r} x^m = \tfrac{1}{2}\alpha((1-m)/2)\alpha = \tfrac{1}{4}\alpha^2(1-m),$$

$$x^k(x^l x^m) = x^k(\tfrac{1}{2}\alpha x^{l+m-p^r}) \ \text{since } l + m > p^r,$$

$$= \tfrac{1}{2}\alpha((1 - l - m + p^r)/2)\alpha = \tfrac{1}{4}\alpha^2(1+k),$$

$$(x^m x^l)x^k = \tfrac{1}{2}\alpha x^{m+l-p^r} x^k = \tfrac{1}{2}\alpha((1-k)/2)\alpha = \tfrac{1}{4}\alpha^2(1-k),$$

$$x^m(x^l x^k) = x^m(\tfrac{1}{2}\alpha x^{l+k-p^r}) = \tfrac{1}{2}\alpha((1 - l - k - p^r)/2)\alpha = \tfrac{1}{4}\alpha^2(1+m).$$

Thus,

$$(x^k, x^l, x^m) - (x^m, x^l, x^k) = \tfrac{1}{4}\alpha^2[(1-m) - (1+k) - (1-k) + (1+m)] = 0.$$

(3.4) can similarly be verified for all cases of (i), (ii), and (iii).

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